AETHER AND MATTER

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AETHER AND MATTER

A DEVELOPMENT OF THE DYNAMICAL RELATIONS OF THE AETHER TO MATERIAL SYSTEMS

ON THE BASIS OF THE

ATOMIC CONSTITUTION OF MATTER

INCLUDING A DISCUSSION OF THE INFLUENCE OF THE EARTH'S MOTION ON OPTICAL PHENOMENA

BEING AN ADAMS PRIZE ESSAY IN THE UNIVERSITY OF CAMBRIDGE

BY

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PREFACE

Aedificium autem hujus universi structura sua, intellectui humano contemplanti, instar labyrinthi est.—F. BACON

THE following Essay was originally undertaken mainly as a contribution towards the systematic theoretical development of the standpoint which considers electricity, as well as matter, to be constituted on an atomic basis.

This is, as regards its general idea, no recent hypothesis. Within ten years of the publication of the fundamental discoveries of Volta, in the year 1800, which first revealed the existence of permanent electric currents, Sir Humphry Davy had been led to maintain the proposition that "chemical and electrical attractions were produced by the same causes, acting in the one case on particles and in the other on masses*," as the outcome of the researches on electro-chemistry which are recorded in his earlier Bakerian Lectures: and a little later the foundation for a complete system of chemistry was sought by Berzelius in a distinction between electro-positive and electro-negative atoms. Although it would be of course wrong to read back our precise modern knowledge into the general views which a survey of the facts impressed on Davy's mind,—just as it would be erroneous to consider his more widely known views on the nature of heat as an anticipation of modern exact thermal theory—yet his striking pronouncements show how

* See Appendix D, *infra*, p. 317.

rapidly the times became ripe for the quantitative electro-chemical investigations of his successor Faraday. Since Faraday's work on electrolysis the notion of the atomic constitution of electrification, in its electro-chemical aspect, has never been entirely absent: it has been insisted on by Maxwell and more particularly by von Helmholtz: it was adapted, in the most natural manner, to the ideas of the Weberian electrodynamics, which treated of moving electric particles: but it is only recently that any efforts have been made towards the development of the Maxwellian aether-theory on that basis.

The form under which the atomic electric theory is introduced in this Essay, originally presented itself—as it happened—in a quite different connexion, in the course of an inquiry into the competence of the aether devised by Mac Cullagh to serve for electrical purposes as well as optical ones. It was found, reasoning entirely from abstract principles, that the only possible way of representing electrification, in an elastic aether, was as a system of discrete or isolated electric charges, constituting singular points involving intrinsic strain in the structure of the medium. If the propagation of disturbances in the aether, with their ascertained optical properties, is to be explained dynamically at all, that is by the interaction of inertia and motion, the elastic reaction of this medium to displacement is almost restricted, as Mac Cullagh showed, to be effectively (even if not fundamentally*) a purely rotational one; an essential confirmation of this rotational character of its elasticity here presents itself in the fact that in a medium

* It is not superfluous to repeat here that the object of a gyrostatic model of the rotational aether is not to represent its actual structure, but to help us to realize that the scheme of mathematical relations which defines its activity is a legitimate conception. Matter may be and likely is a structure in the aether, but certainly aether is not a structure made of matter. This introduction of a suprasensual aethereal medium, which is not the same as matter, may of course be described as leaving reality behind us: and so in fact may every result of thought be described which is more than a record or comparison of sensations.

thus constituted the structure of an atomic electric charge can be directly specified, so far at any rate as is required for a knowledge of its interaction with other electric charges at sensible distance, whereas the absence of any conception of what constitutes electrification had previously been commonly regarded as the fundamental obstacle to the electrical development of aether-theory. The order of ideas thus indicated, when carried through up to its logical extent, involves the explanation of the atomic character of matter itself: matter must be constituted of isolated portions each of which is of necessity a permanent nucleus or singularity in and belonging to the aether, of some such type as is represented for example by a minute vortex ring in perfect fluid or a centre of permanent strain in a rotationally elastic medium. It is thus natural to infer that the ultimate atom of electricity is one aspect of the entity which constitutes the ultimate atom of matter, a conclusion which (foreshadowed as above in set terms by Davy) is almost demonstrated by Faraday's electro-chemical law expressing an exact numerical connexion between them. The question must of course remain open as to whether other forms of activity besides this electrical one can be recognized in the constitution of the atom of matter: as yet nothing seems to have been found in the ascertained types of general physical and chemical phenomena which demands a further amplification, so that any advance in that direction would at present be premature if not gratuitous. For it is to be borne in mind that the proper aim of an atomic theory is not to attempt the impossible task of reducing once for all the whole complex of physical activity to rule, but is rather to improve and connect accepted methods of explanation of the various main regular types of interaction that have been brought to light in this field of knowledge. It is incumbent on us to recognize an aethereal substratum to matter, in so far as this proves conducive to simplicity and logical consistency in our scheme of physical relations, and helpful towards the discovery of hitherto

unnoticed ones; but it would be a breach of scientific method to complicate its properties by any hypothesis, as distinct from logical development, beyond what is required for this purpose. It may therefore be held that, in so far as theories of the ultimate connexion of different physical agencies are allowed to be legitimate at all, they should develope along the lines of a purely electric aether until critics of such a simple scheme are able to point to a definite group of phenomena that require the assumption of a new set of properties and that moreover can be reduced to logical order thereby: a charge of incompleteness without indication of a better way, is not effective criticism in questions of this kind, because, owing to the imperfection of our perceptions and the limited range of our intellectual operations, finality can never be attained.

It appears to be not without value, as regards clearness and definiteness of view, that the conception of an elastic aethereal medium that had been originally evolved from consideration of purely optical phenomena, is capable of direct natural development so as to pass into line with the much wider and more recent electrodynamic theory which was constructed by Maxwell on the basis of purely electrical phenomena,—in fact largely as the dynamical representation and development of Faraday's idea of a varying electrotonic state in space, determined by the changing lines of force.

In the following discussions some care has been taken to trace the connexion with the historical course of physical ideas. While there has been a steady gain in precision, the trend of fundamental physical speculation has always been much the same, and thus does not in its main features pass out of date: but owing to the rapid accumulation of new phenomena and the consequent necessity of formal treatises digesting the state of actual knowledge, there is less and less opportunity to become critically acquainted with the points of view of the original discoverers in physical science, and the general lines of thought in which the definite recorded advances are often only

incidents. The accumulation of experimental data, pointing more or less exactly to physical relations of which no sufficiently precise theoretical account has yet been forthcoming, is doubtless largely responsible for the prevalent doctrine that theoretical development is of value only as an auxiliary to experiment, and cannot be trusted to effect anything constructive from its own resources except of the kind that can be tested by immediate experimental means. The mind will however not readily give up the attempt to apprehend the exact formal character of the latent connexions between different physical agencies: and the history of discovery may be held perhaps to supply the strongest reason for estimating effort towards clearness of thought as of not less importance in its own sphere than exploration of phenomena. Thus for example, the present view of the atomic character of electricity, which is at length coming within the scope of direct experiment, has been in evidence with gradually increasing precision ever since theoretical formulations were attempted on this subject: in fact if the considerations explained below (§ 46) are valid, it is the only view that could logically be entertained without involving either a reconstruction of the accepted basis of physical representation or else an admission of its partial or merely illustrative character.

Some of the following chapters may be regarded as a re-statement in improved form of investigations already developed in a series of memoirs, *Phil. Trans.* A 1894–6–7. They cover only a part of the survey that was there attempted, being concerned mainly with the systematic construction of general electric theory on the basis of intrinsic atomic charges. Judging from some criticisms which that method has attracted, it would appear that misconception has existed owing to difficulty in attaining the point of view, such as may possibly arise from the circumstance that the fundamental concept of the electron did not there present itself at the beginning of the discussion, but was introduced subsequently in an appendix. In estimating the value of an undertaking of this kind, it should of course be

judged as a connected argument. Single departments of electric theory can be, and usually are, more concisely formulated when the nature of their connexion with other departments is not a question of immediate interest. Here the foundation is intended to be definite and universal, thus precluding the adoption of an independent standpoint on each branch of the subject, to be developed as far as it will go, with the help if need be of empirical modification irrespective of the requirements of other branches: and the main question to be determined is not whether the presentation is entirely free from flaw, but whether and in what respects it is an advance sufficient to merit further attention with a view to its improvement. The time has fully arrived when, if theoretical physics is not to remain content with being merely a systematic record of phenomena, some definite idea of the connexion between aether and matter is essential to progress; the discussions which follow are largely concerned with the development of the consequences of one such conception. It seems reasonable to hold that the utility of such a discussion will not be entirely removed should this conception turn out to be practically incomplete, or even incoherent: for it is concerned with a body of ideas relating to the ultimate activity of molecules which is on a different plane from the indefinite notion of mutual forces to which dynamical molecular science has mostly been restricted.

It is recognized of course that every attempt at improvement in scientific exposition must have a limited range, and that the chief critical interest will soon be transferred from what can be explained by any new formulation to what it has not shown itself competent to include. Yet it has turned out, to take a famous instance, to be no insuperable objection to Dalton's chemical development of the atomic theory, that he could form no idea as to how his atoms held each other in combination: although in fact Wollaston, and to some extent Davy, hesitated about accepting the atoms while holding to the laws of combination in definite and multiple proportions that

were suggested by them, because the theory did not explain how an atom is constituted. An instructive illustration of the practical divergence of view that can arise in this manner is afforded by the difference of opinion (cf. Appendix D *infra*) between Newton and Huygens regarding the scope of the law of gravitation.

The general conception of a kinetic molecular structure for matter invites a reconstruction of the theoretical basis of ordinary mechanics itself. The customary development of abstract dynamics rests on the foundation of forces acting between particles so as to satisfy Newton's three laws of motion. The only meaning that can here be assigned to such a particle is an unchanging element of mass whose volume is large enough to contain a vast assemblage of molecules. Even in the hands of Kirchhoff, who is considered to have in his treatment notably broadened the subject, the dynamics of finite bodies is derived through the principle of Least Action from the conception that they are constituted of particles acting on each other in this way; thus in fact adhering to Lagrange's original procedure developed in the year 1760. It seems not too much to say, in the light of molecular science, that such a constitution for a material body is purely imaginary, and even meaningless when applied to such subjects as stress and strain in elastic matter. The real foundation of general abstract mechanics is either the principle of Action in its general form, assumed as a descriptive analytical formulation of the course of phenomena, or else the principle of d'Alembert which is analytically equivalent to it. The latter is doubtless the simpler basis when we are dealing only with elements of mass in the ordinary sense; for it is merely the statement that the irregular or uncoordinated part of the internal motions and strains in a stable or permanently existing element of mass has nothing to do with the changes of configuration of that element considered as a whole; and the preparation for its application consists in the process of picking out and specifying the coordinated parts so far as is needed for

the problem in hand. In electrodynamic problems however the individual electrons in the element of volume come into consideration, at any rate as regards their main division into positive and negative: and then the Action principle, as being the more universal, must almost of necessity be employed. If further we assume that the material molecule is wholly of aethereal constitution, this general dynamical principle of Action must itself be involved (cf. Appendix B) as a direct consequence of whatever scheme of properties is assigned to the free aether, irrespective of special hypothesis: so that, reasoning back from its truth, we may gain some general knowledge as to the nature of the interactions exerted by the ultimate material atoms across the aether. Its significance would then consist, not in any directly ultimate character such as it was originally supposed to possess, but in its being a derived analytical formulation sufficiently comprehensive to cover by itself the domain of mechanical science, as well as (in a minor degree) in its aptitude for analytical transformation to suit whatever aspect of the phenomena is under discussion, after the manner illustrated for example by the analysis of Chapter VI *infra*. In the course of such an abstract development of the dynamics of mechanical systems from the Action principle, the idea of force would be introduced through the coefficients in the completed variation of the Action, to which it is necessary for purposes of physical discussions and comparisons to give a name, that name which is in fact suggested from our sensation of muscular effort. It is possible that these considerations are insisted on in various connexions to an extent that will be tedious*: but they are at

* At one time it was customary to appeal to absolute dynamical principles relating to forces, as the fixed unchanging datum of physical science. The interest in fundamental discussions regarding the mode of formulation of dynamics has however revived in recent years, mainly through the writings of James Thomson, and of Hertz and Boltzmann, and of the school of physicists which advocates restriction to a purely descriptive science of energetics. The conception of the subject that is propounded here is different from the points of view of these writers; while it aims at defining the domain (including all that of steady or very slowly varying states) within which the simpler principles of

any rate not more prominent than the cognate fundamental discussions on convergency and functionality have become in pure mathematical analysis. A field in which these two types of approximation towards ideal exact procedure have something in common is sketched in the analysis of Appendix A: it is now recognized that, in a strictly rigorous presentation of the theory of gravitational and other agencies, it is necessary to inquire (at considerable length) how far a potential is a function that can legitimately be differentiated at a point inside the attracting mass: it is here explained, among other things, that in a physical problem the potential about which these mathematical discussions arise is not the potential of the actual molecular distribution of matter but that of an ideal smoothed-out or continuous distribution, in fact a mechanical* representation which is, only in certain definite respects, equivalent to it. Refinements of this kind are no doubt foreign to an empirical formulation of mathematical physics, where the aim is merely a concise expression of the facts of observation; but it seems not unlikely that in the final theory of the transition from molecular to mechanical dynamics they will be of fundamental import.

The main general principle in this domain, which is considered in some aspects more at length in the memoirs above referred to, is that the mechanical and the molecular properties of a material system, which is not undergoing constitutive change, are independent of each other and not mutually involved: so that the mechanical interactions of a system can be developed independently of a knowledge of its molecular constitution. This principle may even be sufficiently deep-seated to have a bearing on the solution of the philosophical question of ultimate mechanical determinism.

energetics can supply an adequate clue to the course of physical and chemical change.

* Throughout this Essay the term *mechanical* is used in antithesis to *molecular*: *mechanics* is the dynamics of matter in bulk, in contrast with molecular dynamics.

No apology is offered for what may possibly be considered as the non-mathematical character of a considerable portion of the book. The physical hypothesis has been kept intentionally in the foreground, and algebraic results have been where possible translated into their descriptive equivalents. This synthetic course, while more flexible, doubtless makes the exposition more difficult to follow, and apparently less exact, than would be a mathematical development from a system of initial hypotheses, in which attention would be demanded for the analytical processes alone: but on the other hand it exposes the whole situation, and conduces to the direct detection and examination of discrepancies that might possibly otherwise remain latent: it is thus, notwithstanding the somewhat imperfect focussing of the subject, more suitable for a theoretical procedure which is in the constructive stage.

The Essay in its present form was completed at the end of the year 1898, except as regards the Appendices D, E, and F, and the articles and footnotes distinguished by a double asterisk or other mark. In the revision of the sheets before publication the writer has however had the good fortune to obtain the collaboration of his friend Prof. A. E. H. Love, of Oxford, whose acute and vigilant criticism has led to many improvements in the exposition as well as the correction of various mistakes, thereby adding very substantially to the value of the work. For similar most valuable services relating to the latter half of the book, and for the greater part of the list of *corrigenda* —as regards some of which special apology must be made—the writer is indebted to his friend Prof. W. McF. Orr, of the Royal College of Science, Dublin. Notwithstanding much increased confidence in the general validity of the argument, arising from the expert assistance and criticism thus most kindly afforded, many serious defects doubtless remain. Various questions not ripe for final definite treatment, and matters on which opinions can differ, have been passed under consideration: for instance, the discussion on the molecular basis of mechanics

will have largely served its purpose if it draws attention to the point of view. In questions of this kind the most fruitful source of progress is perhaps the process of mutual approximation of different standpoints; any single attempt at effective adaptation of fundamental conceptions must involve detail that is only provisional, and leave points of difficulty unsolved or imperfectly analyzed, while in many cases it will originate more problems than it settles. This latter feature of general theoretical physics cannot in fact be better illustrated than by the original foundation of all such theories as given by Maxwell, which in its broad outlines was the culmination of the greatest advance in modern times: yet it presented itself with so many gaps in its formal development, and raised so many new questions for discussion, that finality with regard to the mode of formulation of the subject is possibly yet far off.

Acknowledgment is due to the officials of the University Press for the exact and obliging manner in which they have executed the printing and corrections.

ST JOHN'S COLLEGE, CAMBRIDGE
Mar. 6, 1900

CONTENTS

CORRIGENDA ✓

Page 11 line 30, **for** $n/2\pi$ **read** $2\pi n$
" 12 " 8, *delete* no. There are viscous tractions which generate heat in the fluid; though they do not affect its motion, they exhaust the energy of the moving solids
" 32 " 25, *after* cases *insert* of limited variation. Cf. p. 276
" 41 " 17, **for** $v^2/V^2 \cos^2\theta$ **read** $v^2/V^2 \sin^2\theta$
" 43 " 3, **for** v **read** v'
" 44 at foot, **for** OA' **read** $O'A$; *delete* $\delta\theta$ in following lines: amend first equation on next page, and change sign of right hand in second and third equations
" 58 at foot, **for** $K\mu/C^2$ **read** $C\cdot/K\mu$, and **for** $(K\mu)^{\frac{1}{2}}/C$ **read** $C/(K\mu)^{\frac{1}{2}}$
" 90 in equation (i) *delete* the fluxion dots: cf. footnote, p. 330
" 104 line 6 from foot, **read** $\gamma(v-\dot{g})-\beta(w-\dot{h})$
" 114 lines 9 and 10, P, Q, R should be cyclically interchanged
" " line 12, multiply by $\frac{1}{2}$
" 121 " 11, **for** K **read** K_1
" 129 lines 20—29, the statement should be reversed: copper filings in a magnetic field alternating as regards intensity but not direction will set themselves *along* the field
" 134 line 4 from foot, **for** c^2/n^2 **read** n^2/c^2
" 135 " 4, **for** $4\pi\mu$ **read** $4\pi\mu\,(n/p_2)^2$; line 7, **for** $2/K\mu$ **read** 2; line 10, **for** only the purely aethereal part **read** the whole of it; and line 14, *after* not *insert* all. In an undamped wave-train the energy is all propagated, up to this approximation: the difference between group velocity and wave velocity involves the theory of dispersion
" 136 **for** last two lines **read** only when K or σ is infinite or else both of them vanish
" 146 last line, *delete* $-$
" 153 lines 8 and 12 from foot, **for** ϵ **read** $\epsilon^{\frac{1}{2}}$
" 154 line 2, **for** $\epsilon^{-1}Q$ **read** $\epsilon^{-\frac{1}{2}}Q$: line 2 from foot, **for** ϵ^{-1} **read** $\epsilon^{-\frac{1}{2}}$: and in last line multiply by $\epsilon^{\frac{1}{2}}$
" 155 " 9 from foot, **for** 4π **read** $4\pi c^2$
" 158 " 12 from foot, **for** $K+2$ **read** $K-2$
" 167 " 15, **for** the second h **read** c
" 174 " 2, and 2 from foot, **for** v **read** v
" 175 lines 1 and 3, **for** t **read** t'

Page 179 line 3, **for** $3\cos^2\theta'$ **read** $\cos^2\theta'$
,, 183 footnote, **for** B **read** C
,, 190 line 7, **for** ϕ/k **read** ϕk^2; and line 8, **for** and **read** though not
,, 198 ,, 2 from foot, **for** c^{-2} **read** c^2
,, 199 lines 10 and 11, **for** a_3 **read** a_3c^2
,, ,, line 15, **for** $+\iota a_3$ **read** $-\iota a_3$
,, 200 on lines 3, 11 and 18, the multipliers before d/dt should be $n\eta n'$ and $n\eta m'$ respectively, where (l', m', n') is the direction vector of the magnetic field
,, 204 the sign of (a_1, a_2, a_3) has been changed in § 132
,, 206 line 9, **for** C **read** C', equal to $C\left(\frac{K-1}{4\pi c^2}\right)^2$; and line 12, **for** C **read** $8\pi c^2C'$, and **for** $)^{\frac{1}{2}}$ **read** $)^{-\frac{1}{2}}$. Cf. also Appendix F, § 5
,, 308 footnote, **for** 1884 **read** 1894
,, 350 **for** $-\Sigma A_{1r}$, $-\Sigma A_{2r}$ **read** $+\Sigma A_{1r}$, $+\Sigma A_{2r}$

ADDENDA AND CORRIGENDA. ✓

p. 2 add to footnote ‡

A treatment of electrons as singularities in the aether has also been developed independently by E. Wiechert 'Die Theorie der Elektrodynamik und die Röntgen'sche Entdeckung,' Schriften d. phys.-ökonom. Ges. z. Königsberg, 1896, and more especially 'Grundlagen der Elektrodynamik' Leipzig (Teubner), 1899.

p. 60 for § 37 line 9 to end read

$$\frac{dR}{dy} - \frac{dQ}{dz} = -\frac{\delta a}{dt},$$

where $\frac{\delta}{dt} = \frac{d}{dt} + v\frac{d}{dx}$; and (P, Q, R) is the electric force acting on the convected charges, being equal to $(P', Q' - vc, R' + vb)$; in which (P', Q', R') represents $4\pi C^2 (f, g, h)$ and is the force straining the quiescent aether. Also $(f', g', h') = \frac{K-1}{4\pi C^2}(P, Q, R)$, as the convection cannot alter the value of K to the first order. These equations are equivalent to the preceding.

p. 86 line 21 add to footnote

More strictly, if the electron were moving with velocity v such that $(v/C)^2$ cannot be neglected, e must be defined as the value of $\int(lf + mg + nh)\, dS$ taken over any surface surrounding it alone.

p. 91 line 29 add footnote

The current then contains magnetic terms: see § 73.

p. 93 add to footnote

The result of not making Φ null, as in the text, would be merely that the electric potential Ψ (§ 58) would be diminished by the amount $-d\Phi/dt$; and the (very minute) distributions of electric charge arising from electrokinetic causes would thus remain undetermined by the theory.

p. 94 add to footnote

L is constant only when the squares of quantities like $\dot{x}/c$ can be neglected. Thus Maxwell's law of pressure of radiation cannot be extended to theoretical discussions involving radiators or reflectors moving with velocity comparable with that of light.

p. 102 for lines 1—7 from foot read

During this transfer the moment of the doublet gradually changes, thus the circuit is not a parallelogram: moreover we must make a correction by superposing on the complete circuit a creation of the doublet in its final position at the end of the transfer and an annulment in its initial position at the beginning of the transfer, in order to obtain the net effect of its transfer and change, free from completion of the ends of the circuit. Combining these creations and annulments for adjacent elementary circuits all traversed in the same element of time, we see that they constitute the polarization current already specified in § 62.

[There will be surface terms outstanding at sudden transitions, which are best evaluated independently by considering the limit of a gradual transition.

This analysis of the electric motions in the individual molecules is necessary for the correct determination of the mechanical forces acting on the system through its electrons (§ 65). But in treating of the electrodynamic propagation we must ultimately replace all magnetism by equivalent aggregate linear electric flux (p. 263) obtaining $\delta/dt\,(f', g', h')$, where δ/dt stands for $d/dt + p\,d/dx + q\,d/dy + r\,d/dz$, as the total current of polarization, which thus represents the time-rate of separation *per unit volume* of the positive and negative electrons in the molecules.]

p. 105 line 4 from foot for dE'/dt read $C^{-2}\,dE'/dt$.

p. 112 after § 70 add a footnote

The potentials, combined in a different manner, may in fact be expressed as directly propagated from the sources of the aethereal disturbance, which are the stationary and moving electrons: see footnote to p. 227, added *infra*.

ADDENDA AND CORRIGENDA.

p. 116 line 7 add footnote

The true current is under all circumstances defined by the flux of electrons across *fixed* interfaces. At the same time it constitutes a volume distribution $(u_1, v_1, w_1)\,\delta\tau$, as *supra* under p. 102.

p. 146 line 11 **for** diminished **read** increased **and for** deducted **read** added.

p. 147 line 4 **for the second** A **read** $-A$.

p. 154 in lines 8, 9 read

; but their level surfaces will linearly correspond to those of

p. 154 line 14 **dele** radial but.

p. 155 line 14 **for** force **read** potential.

p. 157 line 16 **for** force **read** distribution.

p. 176 line 10 **for** charges **read** volume-densities.

p. 201 add at end of § 129

See p. 354.

p. 222 lines 1, 5, 7 from foot, **for** c^2 **read** c^{-2}.

p. 227 line 5 from foot **for** $\frac{3}{4}$ **read** $\frac{1}{4}$.

p. 227 add to footnote

Following Levi-Civita (*Nuovo Cimento*, 1897) the general result may be compactly and instructively expressed as follows. Let (u_1, v_1, w_1) denote the *true* electric current including the equivalent of magnetic whirls, if any, and let ρ be the density of true electrification. Let $u_1(t)$ denote the value of u_1 at time t, so that $u_1(t-r/c)$ denotes its value at time $t-r/c$. Define F_1, G_1, H_1 and Ψ_1 by the relations

$$F_1(t)=\Sigma\frac{u_1(t-r/c)}{r}\delta\tau,\quad \Psi_1(t)=c^2\Sigma\frac{\rho(t-r/c)}{r}\delta\tau.$$

Then the state of the medium is expressible in terms of these *new* potentials by the same equations as give it in terms of the usual vector and electric potentials. These modified potentials here appear as transmitted without loss in symmetrical spherical sheets from the sources of the disturbance, which are the true electric fluxes and electrifications.

Consider in fact an electron e suddenly displaced from A to B. When at rest at A it possessed an intrinsic field of electric force, derived from a potential c^2e/PA: the shift of it sends out an electromagnetic pulse in a spherical sheet: after the passage of this pulse over P the intrinsic electric field at P remains that derived from the new potential c^2e/PB. Then a further displacement of e may follow, producing a pulse which will again alter the intrinsic electric field at P as it passes over that point. By superposing the kinetic electric reaction of the pulse on the intrinsic field of the electron, and summing for all the electrons, the general expression for the electric (or rather aethereal) force at P is immediately obtained; while the magnetic force is the curl of (F_1, G_1, H_1).

At foot of p. 226 the terms involving r^{-3} constitute the statical field of a doublet el; in the problem there considered they should be replaced by that of the single electron, namely by an electric force er^{-2} along r.

p. 297 § 6 for first paragraph read

It has appeared (§ 1) that the result of the unequal drifts of the positive and negative ions, under the action of the electric force, is necessarily an electric current conveyed to an equal extent by positive and negative ions, together with a drift of the electrolyte in mass. In steady electric flow round a circuit made up of various electrolytes there would be a uniform current of this type in each of them, together with a drifting accumulation of the electrolyte continually being neutralized by steady backward diffusion. If a portion of the circuit is metallic, such an accumulation would also be produced in it unless (i) the positive and negative ions or electrons have equal mobilities, or (ii) their mobilities are both so great as to render the accumulation insensible.

p. 332 add footnote to line 5

The nucleus being of unknown constitution, its mobility has to be assumed; that of its strain form is already secured.

Jan. 1901.

RELATIONS OF AETHER TO MATTER

CHAPTER I

INTRODUCTION

THE scheme of this essay may be summarized as follows.

1. In the first section a historical account is given of the progress of experimental knowledge of the influence of the motion of matter through the aether on phenomena directly connected with that medium. For a long period this group of phenomena was purely optical; the principal member of the group was the astronomical aberration of light, and the main interest of the others was centred in their use as tests of theories constructed to account for or explain aberration. The recent development of the connexion between the theories of electricity and optics has added to this domain the class of electro-optic and magneto-optic phenomena, and also various phenomena purely electric. The progress of theory is here passed under review alongside of the progress of experimental knowledge, and some attempt is made to compare the strengths and relative bearings of the positions that have been from time to time taken up by successive investigators. In this review brevity and interconnexion have been chiefly aimed at, for the subject has been historically a rather tangled one owing to the number of writers who have treated it and the variety and isolation of their standpoints: it will be seen that the estimates here given of the relative merits of the various partial theories differ somewhat in character from those generally current. For

references to the earlier historical data Ketteler's treatise* has proved most useful: the principal writings among these, and all the later ones, have also been examined directly.

To this review is appended a general account of wave and ray propagation in moving media, which though originally written independently, necessarily follows very much the same course as the one given in Lorentz's memoir of 1887†.

2. The second section developes the general theory of the relations between matter and aether, which is to form the basis of the treatment of moving material media. In it the electric and optical activity of the matter are assigned to the presence of electric charges associated with the material atoms: the complete scheme of electrodynamic and optical equations is then derived as a whole, on this basis, from the single foundation of the Principle of Least Action. The modifications which arise in the scheme owing to motion of the matter are on this hypothesis directly and definitely ascertainable, on the supposition that the motion of the matter does not affect the quiescent aether except through the motion of the atomic electric charges carried along with it. It appears that the law of the phenomenon of astronomical aberration is fully verified, as are also all the other first order effects—mostly of a null character—of the motion of material systems, which experiment has established. This portion of the subject has been already profoundly treated by Lorentz‡, by an analysis very different from the present one, but with ideas and results that are in the main in agreement with those here arrived at. In the treatment here given, the essential distinction between molecular theory and mechanical theory, and the principles involved in effecting the transition from the former to the latter, are carefully traced.

* 'Astronomische Undulations-Theorie, oder die Lehre von der Aberration des Lichtes,' von Dr E. Ketteler. Bonn, 1873.

† H. A. Lorentz, 'de l'influence du mouvement de la Terre sur les phenomènes lumineuses.' *Archives Néerlandaises* xxi, pp. 103—176.

‡ 'La Théorie Électromagnetique de Maxwell, et son application aux corps mouvants,' *Archives Néerlandaises* xxv, 1892: 'Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern,' Leiden, 1895.

eatment of electrons as singularities in the aether has also been developed independen
E. Wiechert 'Die Theorie der Elektrodynamik und die Röntgen'sche Entdeckung,' Schrifte
phys.-ökonom. Ges. zu Königsberg, 1896, and more especially 'Grundlagen der Elektrodynamik' Leipzig (Teubner), 1899.

3. The third section enters on more speculative ground. It developes the exact consequences, as regards the influence of convection through the aether, which flow from the hypothesis that the atom of matter is constituted of an orbital system of equal primary electric point-charges (or electrons) and of nothing else: or, what comes in certain respects to the same thing in a mode of statement that may possibly be preferred, it assumes that the mass of each sub-atom is proportional to the absolute number of electrons, positive and negative, that it carries, and that the effective interatomic forces are entirely or mainly electric. From this basis a complete formal correlation is established between the molecular configurations of a material system at rest and the same system in uniform translatory motion, which holds good as far as the *square* of the ratio of the velocity of the system to the velocity of radiation. This correspondence carries with it as a consequence the null result, up to the second order, of the very refined experiments of Michelson and Morley on the influence of the Earth's motion on optical interference fringes. The correlation presupposes that the material atoms are independent systems that maintain their relative positions: thus in the simplest case, with which alone we are actually concerned, the material bodies are supposed to be solid, and the influence of the distantly wandering ions, if there are such, that convey electric currents, is left out of consideration as relatively negligible on account of the smallness of their number. In an appendix the mechanism of electrolytic conduction is scrutinized, primarily with a view to drawing conclusions by analogy as to the extent and character of the migrations of the ions in solid conductors: this discussion has however grown altogether out of proportion to its connexion with the present subject, and forms to some extent a connected theoretical account of electrolysis and the voltaic phenomena associated with it, such as the concentration of the electrolyte investigated by Hittorf, the electromotive forces of concentration investigated by von Helmholtz and Nernst, the electric osmosis of Quincke, and the nature of contact differences of potential.

4. It is generally held, chiefly on the ground of Lorentz's

analysis, that the absence of any dependence between the optical rotatory power of quartz and the direction in which the light travels with regard to the Earth's motion, is in discrepancy with theoretical schemes like the present one which consider the Earth to move through the aether without carrying that medium along with it. In the fourth section this question is treated, with results however that prove to be in accord with the facts. As there seems to exist a feeling,—put in evidence by Lorentz's conclusion above mentioned, which asserts a convective influence on rotatory power although he had shown that there was none such on ordinary optical phenomena—that these rotatory phenomena are intrinsically of a class by themselves, a view which may derive strength from their relatively slight or residual character, a discussion of the general nature of the structural and the magnetic rotatory optical properties is given. An attempt to connect rotatory power with density fails, for reasons that are tentatively suggested. It is pointed out that the absence of any convective influence on the rotation affords some independent evidence, in addition to Michelson's result above stated, in favour of the effective validity of the view as to molecular constitution that is considered in the third section.

5. The fifth section treats of the subject of the radiation of material systems, the difficulties of which are not peculiar to any special theory of the connexion between aether and matter. The present theory, which attributes radiation to the oscillatory motions of electrons in the molecule, must give some account of why it is that molecules radiate only when they are violently disturbed; and in particular, which concerns more closely our special subject, why it is that motion of bodies through the aether does not affect the amount or quality of their radiation, except after the merely kinematic manner of the Doppler effect. To carry out this purpose, an expression is found for the train of radiation and of general aethereal disturbance emanating from a single electron moving in any manner.

As connected with the molecular theory, and in fact demanded by it, a discussion is also given of the principles on which optical resolving apparatus is able to decompose the

wholly tumultuous train of disturbance which constitutes ordinary white light into an orderly series of trains of simple harmonic waves. It is held (with Lord Rayleigh) that the train of impulses or vibrations which constitutes the Röntgen radiation would be similarly resolved into simple wave-trains of very high frequencies if we had fine enough apparatus to bring to bear upon it; though the molecular structure of ordinary matter is probably too coarse to be sensibly effective for this purpose. Reasons are given for the view, opposed to what is now sometimes perhaps too generally stated, that counting the number of the succession of interference bands, that can be produced with the light from the whole of a sharp bright line in the spectrum of a gas, enables us to form an estimate of the degree of regularity of the vibrations of the individual molecules which emit the radiation: a result which is of importance for both optical and molecular theory.

SECTION I

CHAPTER II

HISTORICAL SURVEY

6. The phenomenon of the astronomical aberration of light was discovered by Bradley as the outcome of an effort, conducted with unusual care, to detect traces of annual parallax in certain stars which passed near his zenith, and so were amenable to accurate measurement. His observations exhibited displacements in the position of each star, which had the expected period of a year; but instead of being towards the Sun they were towards a perpendicular direction, that of the Earth's motion in its orbit, while they followed the same law of the sine of the inclination as parallax would do. After many attempts to coordinate his observations, the clue to the aberrational method of representing them was suggested to Bradley, it is said, by casual observation of a flag floating at the masthead of a ship; when the ship changed its course, the flag flew in a different direction. The analogy is rather more direct when it is the drift of the clouds that is the object of remark; the apparent direction from which they come is different from the real direction when the observer is himself in motion relative to the air that carries them. So, the observer of the star being in motion along with the Earth, the apparent direction in which the light from the star appears to him to come may be expected to be different from its real direction; thus leading to the usual elementary representation of the

aberration as a phenomenon of relative motion. This explanation is absolutely valid if the light is something which travels in definite rays with finite speed, itself undisturbed by the motion of the Earth: on a corpuscular theory it thus requires that the luminous corpuscles are not sensibly affected by the Earth's attraction: on the undulatory theory it involves either that the luminiferous aether is not disturbed at all by the Earth's motion through it, or else that some special adjustment of its motion holds good which gives the same result. The explanation of the phenomenon of the aberration of light thus immediately opens up the whole question of the disturbance of the aether by the motion through it of material bodies like the Earth, and also of the manner in which the reflexion and refraction of light in our observing instruments is affected by their motion along with the Earth. It is merely one particular result,—more prominent because a positive result—in the field of the mutual influence of aether and moving matter.

7. It occurred to Arago, reasoning on the lines of the corpuscular explanation, that inasmuch as the velocity of light is different in glass from what it is in vacuum, the aberration of its path arising from the Earth's motion would also be different in the glass, and therefore the optical deviation caused by a glass prism would vary according as the light traversed it in the direction of the Earth's motion or in the opposite direction. With the achromatic prism which he employed for testing this conclusion, he calculated that this difference might be as much as a minute of arc. The outcome of the experiments showed no difference at all. Arago worked with star light for which the Doppler effect due to relative motion would make a real difference, excessively minute however and beyond his observational means: with light from a terrestrial source which (as Fresnel remarked) would do equally well for his test, the difference would be absolutely null. The significance of this result, as against the then current explanation of aberration, on corpuscular ideas, was fully realized by Arago: and he communicated the facts to Fresnel with a view to eliciting whether there was anything more satisfactory to be adduced on the basis of the wave theory, which he was then engaged in developing

with the support of Arago's powerful advocacy. The phenomenon to be accounted for was that the motion of the Earth does not affect the laws of reflexion and refraction of light. In Fresnel's reply*, which is one of the fundamental documents on the present subject, he pointed out that a simple answer was possible, namely to assume that the surrounding aether is carried along completely by the Earth so that all relative phenomena would be the same as if the Earth were at rest: but he went on to say that this view could not be entertained on account of the facts of astronomical aberration, of which he could form no intelligible conception except on the hypothesis that the aether remained absolutely stagnant as the Earth moved through it. On this latter hypothesis the velocity of light outside a transparent body must have the normal value: and it was an easy problem to find whether it was possible for any law of modification of the velocity of light inside the body, arising from its motion, to make the laws of refraction and reflexion relative to the moving body the same as for matter at rest, as Arago's experiment required. It appeared that there is such a law, the conditions being all satisfied if the absolute velocity of light inside a transparent medium of index μ is increased by the fraction $1-\mu^{-2}$ of the velocity of the medium resolved in its direction. This supposition, adopted on the above grounds by Fresnel, keeps the paths of the rays relative to the moving bodies unaltered, and at the same time satisfies the facts of aberration. The attempt made by Fresnel to provide for it a dynamical foundation suffers from the same kind of obscurity as did his later dynamical theory of crystalline refraction: and though the subsequent views of Boussinesq and Sellmeier, on the part played by the matter in the mechanism of refraction and dispersion, allow a valid meaning to be read into Fresnel's explanations, yet they perhaps form no very essential part of his achievement in this field. Afterward Sir George Stokes showed in detail that Fresnel's hypothesis not only left the relative paths of rays unaltered, but the phenomena of interference as well, some of which had been urged against it by Babinet.

* See Appendix D.

The result obtained by Arago suggested a wide field of experimental inquiry as to whether other optical phenomena as well as refraction were independent of the direction of the Earth's motion through space. In most cases the experimental test is very precise and delicate; for the apparatus exhibiting the optical effect has only to be installed in the most sensitive manner possible, and note taken as to whether the gradual change of absolute direction of the light passing through it, arising from the Earth's movement of rotation, causes any diurnal inequality in the results. The negative results of theory have gradually been extended, by special investigations, to other optical phenomena, such as dispersion and crystalline interference, as these were successively found by experiment to be uninfluenced by the Earth's motion in space. It will be seen that the modern or electric view of the aether supplies a succinct dynamical foundation for the whole matter.

8. Long before Arago's time it had occurred to Boscovich, reasoning from Bradley's original point of view, that inasmuch as the velocity of light in water is different from what it is in air, the aberration produced by the Earth's motion in the apparent path of a ray travelling through water should be different from the normal astronomical amount: he suggested the use of a telescope with its tube filled with water to find out by star observations whether this is the case, in the expectation that the line of collimation would be different, in order that the relative rays in the water should focus on the cross-wires, from what it would be if the interior of the tube contained only air. In recent times Sir George Airy has actually had such an instrument temporarily installed at the Greenwich Observatory: he has found that observations with it, continued over a considerable time, gave the ordinary value of the constant of aberration, the different aberration of the ray in water being thus compensated by a modification of the ordinary law of refraction on the passage of the light into that moving medium. This experiment had already been discussed by Fresnel in his letter to Arago, with the remark that there is no occasion to complicate the result by aberration, as a terrestrial object might equally well be focussed on the cross-wires of the instrument

placed transverse to the direction of the Earth's motion; and that the observations might even be carried out by a microscope, reversible by hand, sighted on a bright point attached to its frame. It was pointed out by Fresnel that Wilson had shown that on the corpuscular theory no effect was to be expected: and he added a demonstration that the same was the case on his own view of the undulatory theory, thus predicting on a *consensus* of all points of view a negative result.

9. The theory was next taken up, from the undulatory standpoint, by Cauchy, who preferred the first of Fresnel's alternatives, that the Earth in its orbital motion pushes the aether in front of it so that the portions near the surface travel along with the Earth, as he was unwilling to admit that the heavenly bodies could move through the aether without disturbing it at all. He pointed out that astronomical aberration was then to be explained, not probably by any effect of changed aethereal elasticity or inertia, but merely by a kinematic slewing round of the advancing wave-fronts (or rather absence thereof) owing to the translatory motion of the medium in which the waves are propagated. The disturbance of the aether itself owing to the motion of the Earth he was prepared to regard as the source of the electric and cognate phenomena associated with that body. This mere preference of Cauchy's did nothing towards removing Fresnel's difficulty as to how such a motion of the aether is to be imagined as exactly adjusted so as to involve the correct amount of aberration in accordance with Bradley's law: but the view subsequently became a real theory in the hands of Sir George Stokes. That physicist had just had cause, in his hydrodynamic researches, to analyze the differential change of form, arising from the state of motion in a fluid medium around any given point, into pure strain made up of three superposed elongations, combined with pure rotation: and it became clear that if the latter component is absent in the aether, so that the motion of the aether is differentially irrotational, the advancing wave-fronts will not be slewed round at all, and therefore the waves will travel through space in straight lines as if the aether were at rest. When these rectilinear waves get into the region of aether imme-

diately around the observer, which is carried on with him, they will be affected relative to him with the full aberrational change of direction arising from his motion, just as a moving corpuscle would be. Now the irrotational quality of aethereal motion thus pointed to, is, by Lagrange's fundamental hydrodynamical theorem, the characteristic of the motion of frictionless fluid which has been originally at rest: thus the material for a physical theory lies at hand. The aether is, as regards slow motions in bulk, simply assumed to have the properties of frictionless continuous fluid substance, while for the excessively rapid small vibrations of light it has solid elastic quality. The question remained how far these two sets of qualities can coexist in the same medium: an affirmative answer was defended, or rather illustrated, by an objective appeal to the actual properties of a substance such as pitch, which flows like water if sufficient time is allowed, while at the same time it can be moulded into an efficient tuning-fork for small vibrations as frequent as those of sound. There appears to be good ground for demurring against the mutual consistency of the properties imputed to a simple, permanent, and flawless medium like the aether being settled by an appeal to the approximate behaviour of a highly complex and viscous body like pitch: the principle that is involved can however be expressed in a purely abstract manner. If any term in the analytical dynamical equations of the aether is made up of two parts, so as to be of type such as $au + bd^2u/dt^2$ where u represents displacement, then when b is very small compared with a the first part au will practically represent the term for slow motions, while on the other hand for simple vibratory motion of excessively high frequency n, d^2u/dt^2 being then equal to $(n/2\pi)^2 u$, the second term is the all-important one. The objection to this kind of explanation, which substitutes a very close approximation for the exact term, is that we have actually to provide in the aether for a transparency which is adequate to convey the light of the most distant stars, which points rather to exact abstract mathematical relations than to complex and approximate physical laws of elasticity.

10. At the same time Sir George Stokes expressed his

belief that it would not do to actually take the aether to be an ordinary fluid, on the ground that this ideal motion of irrotational quality would then be unstable. In a subsequent note (*Phil. Mag.* 1848) he advanced as proof of this instability the fact that the mathematical solution for the *steady* motion of a sphere through a viscous fluid, which he had just obtained, is the same however slight may be the degree of viscidity of the fluid. Now an irrotational motion calls out ~~no~~ viscous reaction throughout the mass,**) and therefore satisfies the conditions of viscous as well as of perfect flow: but there is one circumstance which destroys its claim to be a solution in the former case, namely the presence of slip at the surfaces of the solids. If the surfaces of the solids were ideally frictionless this would not matter: but if when the irrotational flow has there been fully established, the actual frictional character of the surface were restored, laminar rotational motion would spread out from each surface in the same manner as heat would spread out by diffusive conduction from a hot body, until a new state of steady motion would supervene. The solution of Stokes shows (as is also clear from general principles) that however small the viscosity, this new steady state is wholly different from the ideal irrotational steady state belonging to mathematical absence of viscosity and friction: and it might appear to follow that this state is, not precisely an unstable one, but rather one which could not exist at all in the fluid. The term unstable is however appropriate because, if the solids are impulsively started into their steady state of motion, the initial state of motion of the fluid will (*assuming* that there is no such thing as impulsive friction*) be the irrotational one, which will *gradually* be transformed by diffusion of vortex motion from the surfaces at a rate which is the slower the less the viscosity of the fluid. This conclusion follows as a special case of Lord Kelvin's general dynamical principle that when a material system is impulsively set into motion by imposing given velocities at the requisite number of

* The direct proof from the hydrodynamical equations is not however limited in this way, if the law of impulsive viscosity may be assumed to be linear.

**) There are viscous tractions which generate heat in the fluid; though they do not affect its motion, they exhaust the energy of the moving solids.

points or surfaces, (namely in this case given component of velocity normal to the boundaries) the state of motion instantaneously assumed by it is that one for which the kinetic energy is least, which is easily shown to be the irrotational one in the case of a liquid.

As Sir George Stokes was not disposed to admit that the aether could pass freely through the interstices of material bodies in the manner required by Fresnel's views, and as any other theory of its motion which could be consistent with the fact of astronomical aberration required irrotational flow, an explanation of the limitation to that flow had, he considered, to be found. He pointed out that the existence of tangential stress depending not alone (like viscosity) on relative velocities, but also (like elastic stresses) on relative displacements, would make the flow irrotational; for any deviation from irrotational quality would now be propagated away not by diffusion but by waves of transverse displacement, and the coefficient of the elastic part of the force, and consequently the velocity of this propagation, may be assumed so great that the slightest beginning of rotational motion is immediately shed off and dispersed. This chain of argument, that motion of bodies disturbs the aether, that aberration requires the disturbance to be differentially irrotational, that this can only be explained by the dispersion of incipient rotational disturbance by transverse waves, and further that radiation itself involves transverse undulation, he regards as mutually consistent and self-supporting, and therefore as forming distinct evidence in favour of this view of the constitution of the aether*. The coexistence of fluidity on a large scale with perfect elasticity on a small scale he illustrates by the ordinary phenomena of pitch or glue, passing on to a limit through jellies of gradually diminishing consistency until perfect fluidity is reached: the chief difficulty here is (as already mentioned) that absolute mathematical

* It would thus appear that the slip at the surface of the moving solids, which is offered as a decisive objection to Stokes' view by Lorentz, is not really fatal to such a view of aberration, taken by itself, except in so far as it leads to continual radiation from the surface of the moving body and therefore to resistance to its motion.

transparency of the aether is replaced by approximate transparency, such as would involve ultimate decay of all structures existing in it.

11. The problem of the relative motion of the Earth and the aether was treated by Clerk Maxwell in 1867, in a letter to Sir W. Huggins which has been incorporated in the fundamental memoir of the latter on the spectroscopic determination of the velocity of movement of stars in the line of sight*. It is there pointed out that there are two independent subjects for examination. The Doppler alteration of the period of the light from a star is quite definite, and independent of the special details of the form of undulatory theory that may be adopted. But there is a second question as to whether the index of refraction depends on the orientation of the ray with reference to the direction of the Earth's motion, in which the observer and all his apparatus participate: this involves the physical nature of the undulations: here, as Fresnel had already remarked, the sources of light may just as well be terrestrial as astronomical. According to Arago's original experimental result, which had been closely tested by a more delicate arrangement by Maxwell himself working with homogeneous light, some years before this time, there is no influence on the index of refraction arising from the Earth's motion. As refraction depends solely on retardation in time owing to the smaller velocity of propagation in the refracting medium, the relative retardation must therefore be unaltered by the Earth's motion. If V be the velocity of a ray in air, and v the velocity of the aether in air relative to the observer, and if V' be the velocity of the same ray in a dense medium and v' the velocity of the aether in that medium relative to the observer†, then across a thickness a of this medium the light is retarded with respect to air by a time

$$\frac{a}{V'+v'} - \frac{a}{V+v},$$

* *Phil. Trans.* 1868, p. 532.

† This is Maxwell's phrase, no doubt interpreting Fresnel: on a wider and more modern view v' is the amount by which V' is altered owing to the motion of the aether relative to the medium.

which is equal to

$$\frac{a}{V}\left\{\frac{V}{V'}\left(1-\frac{v'}{V'}+\frac{v'^2}{V'^2}+\dots\right)-\left(1-\frac{v}{V}+\frac{v^2}{V^2}+\dots\right)\right\},$$

that is to

$$\frac{a}{V}\left\{\frac{V}{V'}-1-\frac{v'}{V}\left(\frac{V^2}{V'^2}-\frac{v}{v'}\right)+\frac{v'^2}{V^2}\left(\frac{V^3}{V'^3}-\frac{v^2}{v'^2}\right)+\dots\right\}.$$

As this is, by the experimental evidence, to be independent of v and v' to the first order, we must have $v'/v = V'^2/V^2$; or, expressed in words, the effects of the Earth's motion on the velocities of the ray relative to the observer in the two media are proportional to the squares of the ray-velocities for the ray under consideration. In the moving refracting medium the absolute velocity of the ray is therefore increased by $v - v'$, that is by $v(1 - V'^2/V^2)$, where v is the velocity of the medium in the direction of the ray. When the medium is isotropic, V/V' is equal to the index of refraction μ, thus the alteration of the velocity is $v(1-\mu^{-2})$, as Fresnel originally found. (p. 8)

According to the ideas underlying Fresnel's general optical theory, refraction depends on change of density of the aether. Thus the density of the aether in the refracting substance would be proportional to μ^2: and if the aether is imagined as flowing across the refracting substance in its relative motion, its velocity in that substance must by the equation of continuity be μ^{-2} of its velocity in air outside. Thus on the hypothesis that the change of velocity is solely due to a convection by the moving aether, we are led from Fresnel's general notions to the same law as Arago's experiments demands*.

In the same place Maxwell remarks on the great instrumental difficulty, and also the absence of confirmation, of the experiments of Fizeau and Ångström indicating displacement of the plane of polarization by passage through a pile of glass plates and by diffraction respectively, depending on the orientation of the apparatus with regard to the direction of the Earth's motion.

* This remark is given by Maxwell. I do not find it in Fresnel's letter to Arago, but it occurs in part in the paper by Sir G. Stokes, *Phil. Mag.* 1846.

12. It results from very various experimental investigations some of which are mentioned above that, with a very doubtful but unique exception in the case of Fizeau's experiments on piles of glass plates, the most varied optical phenomena, whether of ray paths or of refraction, dispersion, interference, diffraction, rotation of plane of polarization, have no relation to the direction of the Earth's motion through space, though for many of them the test has been made with great precision. The most obvious conclusion from this *consensus* of evidence taken by itself would be the view that the Earth's motion carries the aether completely along with it, and that all the relative optical and other phenomena are therefore just the same as they would be with both the Earth and the aether at rest. Such a view is also very temptingly suggested by the absolutely negative result, up to the second order, of the Earth's motion on the Michelson interference experiment. If then we could assume that the Earth's motion produces flow, differentially irrotational according to Sir George Stokes' criterion, in the surrounding aether, but such that in all regions near the Earth's surface, up to the greatest distance at which we can explore, the aether is practically carried along bodily with the Earth, the requirements both of astronomical aberration and of the mass of negative optical results would be fully satisfied. But here we are met by various difficulties. If we assume that the aether around the Earth near its surface is carried on by the Earth as that body traverses its orbit, and also assume that at a great distance the aether is at rest, these states of motion cannot be connected without discontinuity by any possible irrotational motion of the intervening aether. The irrotational motion set up by the motion of the Earth and the surrounding shell of aether, supposed attached to it, is the same as would be set up by a moving solid in ideal frictionless liquid: the continuity of normal flow can be preserved, but there must be tangential discontinuity (slip) either at the boundary of the solid or somewhere else: this is the case whether incompressibility of the aether is assumed or not, the two sets of conditions continuity of normal flow and continuity of tangential flow being more than can be simultaneously

satisfied. This way of surmounting the discrepancies is therefore, on the very threshold of our present wider survey, illusory. Were it not so, it would only be necessary to proceed a step further in order to encounter fresh difficulties. If the aether were carried on bodily by the Earth, we must assume that the aether very near a mass moving along the Earth's surface is at any rate partially carried along by that mass. This point has been tested directly with great precision by Lodge†, who tried to detect whether the aether between two whirling steel discs partook to any extent in their motion; and the result has been decisively negative. The only possibility of escape from this result, that the aether is not carried on by the Earth's motion, would be in an assumption that the large mass of the Earth controls wholly the motion of the aether in its neighbourhood somehow as it does gravitation, so that the smaller mass of the rotating discs is inoperative in comparison. In any case the former difficulty remains decisive: we might indeed be tempted to replace the absolutely irrotational motion of the surrounding aether, involving surfaces of slip, by very slightly rotational motion such as would evade all tangential slip: but the law of astronomical aberration would thereby be upset, since the smaller the rotation thus imposed the greater the distance to which it must extend, while the resulting aberration is proportional to these quantities jointly*. A hypothesis that would allow the aether to be moved in any degree by material bodies passing across it thus has small chance of correspondence with the body of ascertained optical facts.

We are therefore thrown back on Fresnel's view that the aether is not itself set in motion by the movement of material systems across it, or, in terms of the *simile* of Young, that it passes through the interstices of material bodies like the wind through a grove of trees.

† 'Aberration Problems' *Phil. Trans.* 1893 A.

* It has been suggested by Des Coudres, as a way out of the difficulty, that the aether is possibly subject to gravity: but that would merely produce a balancing hydrostatic pressure without altering the irrotational character of the motion.

13. The next important theoretical contribution to our subject is implicitly contained in §§ 600, 601 of Maxwell's "Treatise on Electricity and Magnetism" (1872). It is there verified, by direct transformation, that the type of the equations of electromotive disturbance is the same whether they are referred to axes of coordinates at rest in the aether or to axes which are in motion after the manner of a solid body. The principle is here involved, as FitzGerald† was the first to point out, and as was no doubt in Maxwell's own mind considering his recent occupation with the subject, that in treating of the electromotive disturbance which constitutes light we are permitted to make use of axes of coordinates which move along with the Earth without having to alter in any way the form of the analytical equations. This statement covers as a special case Bradley's law of astronomical aberration. It also directly includes in its entirety the principle of Arago and Fresnel that the laws of geometrical optics are not affected by the Earth's motion: it ought therefore to involve as a consequence Fresnel's expression for the change of velocity of radiation produced by motion of the material medium which it traverses. The latter question was examined directly from Maxwell's analytical equations by J. J. Thomson* with a result different from Fresnel's, namely, that the acceleration of velocity is always half that of the moving material medium, being the same for all kinds of matter. This discrepancy is one of several which indicate that for extremely rapid disturbances like optical waves, the analytical scheme of Maxwell does not sufficiently take into account the influence of the material medium on the propagation. A contradiction of some kind is also suggested by the circumstance that Maxwell's theorem does too much by making the optical properties independent of uniform velocity of rotation of the material medium, as well as of uniform velocity of translation; we shall see (§ 23) that the possibility of exact independence in both respects is negatived by the general nature of rays. The necessary amendment of the scheme of Maxwell has been independently arrived at by more than one writer, but somewhat earliest in point of time

† *Trans. Royal Dublin Society*, 1882.

* *Proc. Camb. Phil. Soc.* v., 1885, p. 250.

by H. A. Lorentz; and it involves the general electrodynamic considerations, including the discrete distribution of electricity among the molecules of matter, on which the present essay is based. Shortly after Lorentz the subject was taken up from a similar point of view by von Helmholtz, primarily in relation to the theory of optical dispersion: but his equations, derived from a difficult abstract procedure in connexion with attempted generalizations of the principle of Least Action, were soon found to be at fault in the matter of moving media just as much as was the original scheme of Maxwell. Possibly thereby incited, von Helmholtz considers directly the question of motion of the aether in his last published memoir: he finds, as Hertz had done some years before, that the mechanical force as given by the equations of Maxwell cannot by itself keep the aether in equilibrium if we suppose this force to act on it as well as on matter: and on the assumption that the aether is fluid as regards movements arising from extended disturbance, and of very small density, he obtains differential equations for the determination of the steady state of aethereal motion that must on that hypothesis exist in an electrodynamic field. The existence of any finite motion of this sort, unless it is very minute, has been negatived by the elaborate experiments of Lodge, and also by more recent observations on the same plan by Henry and Henderson which were inspired from von Helmholtz's theory.

14. Quite recently a general summary of the state of the question of the mutual relations of aether and moving matter has been published by W. Wien*, as a guide to a discussion of the subject at the annual meeting of the German Scientific Association. He there works out some special cases of von Helmholtz's theory just mentioned, arriving at the result that if the density of the aether is absolutely null there can exist a steady translational motion of electric charge through the aether which will not involve any disturbance of that medium, while if the density is very small the disturbance thus involved will be very slight: but in motions not steady, for example the uniform separation of the components of a stationary electric

* *Wied. Annalen* lxv., July 1898.

doublet, infinite velocities of disturbance of the aether will enter at the very beginning of the motion, so that the steady state cannot be originated.

On the other hand, the abstract theory to be here given may be translated into a concrete scheme which identifies electrodynamic energy with the translatory kinetic energy of the aether considered as possessing inertia: to make the aether remain practically quiescent under all conditions it is then necessary and sufficient to take its inertia to be sufficiently great: in fact if this were not secured, the electrodynamic equations instead of being linear would involve the very great complication of non-linear terms with which we are familiar in theoretical hydrodynamics.

In summing up at the end of the above-mentioned essay, Wien formulates three outstanding objections to the hypothesis of a quiescent aether;

(i) the observed absence of any magnetic effect of the motion of electrically charged bodies carried along by the Earth,

(ii) the absence of any influence of the Earth's motion on the optical rotatory property of quartz,

(iii) Fizeau's experiment, in which he found some evidence for changes, arising from the Earth's motion, in the displacement of the plane of polarization of light produced by passage through a pile of glass plates.

As regards these objections, the first appears to be a point in favour of the theory instead of against it (§ 40 *infra*): the second is based on a theoretical investigation of Lorentz, which appears to be at fault (Ch. XIII.) so that the result is again in favour: while the conclusion in the third case was regarded as doubtful by Fizeau himself on account of the extreme difficulty experienced in excluding disturbing causes (a doubt which has been shared by most authorities who have since examined the matter, including Maxwell and Rayleigh), and the experiment has not been repeated.

Electrodynamic view of the Aether.

15. The astronomical aberration of light is one of the small group of phenomena in which the reactions between matter and aether depend sensibly on the state of motion of the matter. Disturbances originated in the aether are equalized and smoothed out with such great speed, that the aether-field around a body, which is moving with any attainable velocity, is practically at each instant in the same equilibrium condition as if the body were at rest: it is therefore only in the case of very rapidly alternating phenomena such as radiation that there is any practical occasion to pass beyond a mere theory of convection of aethereal effect along with the molecules of the matter. It is owing to this circumstance that the electrodynamic theories of Ampère and Weber represented so well the whole range of phenomena then open to experiment, even to the extent of giving in Kirchhoff's hands the correct velocity (that of radiation) for the transmission of electric waves of very high frequency guided along a wire: and that, as regards the deeper questions of propagation of electric effect in time, theory has been, chiefly in Maxwell's hands, uniformly so far in advance of the means of verification.

The logical validity of the older electrodynamics was confined to systems of uniform currents streaming round closed paths: and all investigations purporting to deduce from experimental data expressions for the electromotive forces induced in open circuits, or for mechanical forces acting on separate portions of circuits carrying currents, were necessarily illusory from the fact that such portions were practically unknown as separate independent entities. The new departure instituted by Maxwell came, when expressed mathematically, to a statement that dynamically all electric discharges are effectively of the nature and possess the properties of systems of closed currents, being completed when necessary by so-called displacement-currents in free space and in dielectric media; in fact that the consideration of the electrodynamics of unclosed circuits never arises. That theory, as left by its author, works out by adapting the established Amperean theory of closed

currents to the new ideas; and there still remains the same ambiguity in respect to mechanical forces on portions of flexible or extensible current-circuits*. There is however no ultimate ambiguity as regards electromotive phenomena in bodies at rest, the equations of the theory sufficing to eliminate the arbitrary element that initially must be introduced, and thus to give a definite determination of the electric force at each point of space. New difficulties, of practical importance only in the theory of radiation, occur when the material medium which carries the current is in motion: and the theory for that case was left in the form of a first approximation, which assumed that the aethereal disturbance was simply convected by the moving matter, that being amply sufficient for ordinary electrodynamic applications. The dynamical methods were however sketched by Maxwell which would have to be employed to work out a more definite scheme of the relation of aether to the matter at rest in it or moving through it: and quantities of dynamical origin or suggestion, such as the vector potential of electric currents, which have sometimes been considered so great a complication by subsequent writers as to justify their summary abolition, turn out in fact to be of the essence of a more thorough analysis.

16. The dynamical scheme which thus in Maxwell's hands furnished a formulation of the electrodynamics of material systems *at rest* in the aether, completely effective except as regards the material mechanical forces acting on the matter carrying the currents, was one of continuous differential analysis: the matter was taken as simply modifying, where it existed, the effective constants in the formula for the spacial distribution of electric energy: when the aether did not move with any finite speed or the matter move across it, there was no pressing occasion to separate the energy into a part belonging to and propagated by the aether and a part attached to the molecules of matter. The theory, at the stage at which it was left by Maxwell, being a theory of complete electric circuits,

* Cf. *Phil. Trans.* 1895 A, pp. 697—701, for a demonstration that the ponderomotive forces cannot be directly deduced from a single energy-function without the aid of molecular analysis.

the total current was a continuous streaming flow; there proved to be no necessity, in the case of systems at rest, for keeping distinct the current of conduction, the current arising from changing electric polarization in a dielectric substance, and the displacement current belonging to free aether apart from matter altogether: the only hypothesis he required was that there is an aethereal current of such amount as to complete into a single circuital stream all the types of true electric flux which are associated with matter. These distinctions however become essential as soon as the theory is to take cognizance of the motion of the matter, especially in the domain of radiation where a mere equilibrium theory, contemplating the convection unaltered of its electric field along with the matter, is not a valid approximation. Then convection, relative to the aether, of electric charge and of dielectric polarization, contributes to the total current, as well as the change of aethereal elastic displacement and of material polarization. The problem thus presents itself in the form of two media, the aether and the matter, each with its own motion, but both occupying the same space; and some idea has to be formed of the interconnexions by which they influence each other. If we treat them both by the methods of continuous analysis, the only way open is to assume the most general linear relations between the two sets of variables representing the properties and states of the two media, and subsequently try to reduce the generality by aid of experimental indications. This is a well-tried course of procedure in abstract physics, and has been very effective under simpler and more easily grasped conditions: but even if successful it could hardly help us to mentally realize the connexion between aether and matter, while on the other hand the philosophical objections to filling the same space with several different media have been widely felt and emphasized.

17. Possibly the only sound procedure is the one which recommends itself on purely philosophical grounds. From remote ages the great question with which, since Newton's time, we have been familiar under the somewhat misleading antithesis of contact *versus* distance actions, has engaged speculation,—how it is that portions of matter can interact on each

other which seem to have no means of connexion between them. Can a body act where it is not? If we answer directly in the negative, the spacial limitations of substance are to a large extent removed, and the complication is increased. The simplest solution is involved in a view that has come down from the early period of Greek physical speculation, and forms one of the most striking items in the stock of first principles of knowledge which had been struck out by the genius of that age. In that mode of thought the ultimate reality is transferred from sensible matter to a uniform medium which is a *plenum* filling all space: all events occur and are propagated in this *plenum*, the ultimate elements of matter consisting of permanently existing vortices or other singularities of motion and strain located in the primordial medium, which are capable of motion through it with continuity of existence so that they can never arise or disappear. This view of physical phenomena, which was no doubt suggested by rough observation of the comparative permanence and the mutual actions of actual whirls in water and air, was quite probably, even at that time, not the mere idle philosophizing which has sometimes been supposed. It at any rate involves the fundamental consequence that the structure of matter is discrete or atomic—that contrary to *à priori* impression matter is not divisible without limit: and it perhaps enables us to form some idea of the line of development of those views on the constitution of matter which, as Democritus and Lucretius described them, were considerably ahead of anything advanced in modern times until the age of Descartes and Newton. The same doctrine was probably the ideal towards which Descartes was striving when he identified space and matter, and elaborated his picture of the Solar System as a compound vortex. In Newton's cautious hands, the relation of material atoms to aether is not dealt with: his establishment of an exact law of gravitation indeed originated the school of action at a distance, which held bluntly that matter *can* be considered as acting where it is not, and whose influence lasted throughout the seventeenth century through Boscovich and the French astronomers and mathematicians, until the time of Faraday. This doctrine of the finality

of action at a distance was however strongly repudiated by Newton himself, and hardly ever became influential in the English school of abstract physics represented by investigators of the type of Cavendish and Young. More recently, the following out into modern developments of the mere idea of continuous transmission of physical actions gained for Faraday a rich harvest of fundamental experimental discoveries: while the general results obtained by von Helmholtz in the abstract theory of fluid motion have enabled Lord Kelvin to reconstruct on a precise scientific basis the notions of Leucippus and Descartes on the relation of matter to aether**.

Meanwhile, irrespective of such general cosmical views, the development of electrical theory itself has been steadily tending to an atomic standpoint. It has been noted by Maxwell, and was afterwards very fully enforced by von Helmholtz, that the interpretation of Faraday's quantitative laws of electrolysis could only be that electricity is distributed in an atomic manner, that each atom of matter has its definite electric equivalent, the same for all kinds of atoms: and even the expressive phrase "an atom of electricity" was imported into the theory by Maxwell. The only difficulty in this mode of formulation related to the mechanism of transference of these atomic charges or electrons from one molecule of matter to another. The order of ideas to be presently followed out will however require us to hold that the atomic charge is of the essence†† of each of the ultimate subatoms, or as we may call them *protions*, of which an aggregation, in stable orbital motion round each other, go to make up the ordinary molecule of matter: so that the transference of electric charge will involve transference or interchange of these constituent protions themselves between the molecules, that is it will always involve chemical change, as Faraday held on experimental grounds must be the case.

** Cf. Appendix D. It may well be that too favourable a view is taken in the text of the earlier physical atomic theories, which up to the period of Lord Kelvin's vortex atoms could only have been hypothetical speculations.

†† Cf. Sir Humphry Davy, in Appendix D.

18. The fluid vortex atom of Lord Kelvin faithfully represents in various ways the permanence and mobility of these subatoms of matter: but it entirely fails to include an electric charge as part of their constitution. According to any aether-theory static electric attraction must be conveyed by elastic action across the aether, and an electric field must be a field of strain: hence each subatom with its permanent electric charge must be surrounded by a field of permanent or intrinsic aethereal strain, which implies elastic quality in the aether instead of complete fluidity: the protion must therefore be in whole or in part a nucleus of intrinsic strain in the aether, a place at which the continuity of the medium has been broken and cemented together again (to use a crude but effective image) without accurately fitting the parts, so that there is a residual strain all round the place.

The assumption of elasticity of some kind in the aether is of course absolutely essential to its optical functions: and the elucidation from the optical phenomena, as a purely abstract problem in analytical dynamics, of the mathematical type of this elasticity, was acomplished in 1839 by MacCullagh* in an investigation which may fairly claim to rank amongst the classical achievements of mathematical physics. The type of elasticity which he arrived at was one wholly rotational, so that the aether would be perfectly fluid for all motions of irrotational type, but would resist elastically, by a reacting torque, any differential rotations of the elements of volume, somewhat after the manner that a spinning fly-wheel resists any angular deflexion of its axis. Here then we have the specification of an ideal medium that would behave as a fluid to solid bodies moving through it, because its irrotational motion would be precisely the same as that of a fluid in the corresponding circumstances: it would not resist the motion of such solids any more than the aether resists the motion of the heavenly bodies or of material masses generally: moreover vortex rings could permanently exist in it and persist according to the well-known laws of abstract hydrodynamics. But these tempting

* 'An Essay towards a Dynamical Theory of Crystalline Reflexion and Refraction,' *Trans. R. I. A.*, vol. xxi: *Collected Works*, p. 145.

indications must be put aside in favour of a track lying in a rather different direction, the ultimate element of material constitution being taken to be an electric charge or nucleus of permanent aethereal strain instead of a vortex ring.

A view of the constitution of matter, which proves to be sufficient over an extensive range of physical theory and must not be made any more complex until it proves insufficient in some definite feature, asserts that the molecule is composed simply of a system, probably large in number, of positive and negative protions in a state of steady orbital motion round each other. Nothing has yet been done directly to examine how wide a field of possibility of different types of molecules and molecular combinations is thus opened up: but it is easy to recognize that the range is more extensive than would be offered by a Boscovichian system of attracting points, or of attracting polar molecules as in A. M. Mayer's illustrative experiments with magnetic elements, or by fluid vortex rings. Thus for example a system of electrons ranged along a circle, and moving round it with the speed appropriate for steadiness, constitutes a vortex ring in the surrounding aether: it will therefore enjoy to some extent the well-known wide limits of stability of such a ring*: and the stability will probably be maintained even when there are only a few electrons circulating at equal intervals round the ring. Again, a positive and a negative electron can describe circular orbits round each other, stable except as regards radiation, thus forming a simple type of molecule devoid of magnetic moment: or again, we might have a ring formed of electrons alternately positive and negative. And moreover we may imagine complex structures composed of these primary systems as units, for example successive concentric rings of positive or negative electrons sustaining each other in position.

The duality arising from the assumption of two kinds of electrons, only differing chirally so that one is the reflexion of

* It is here implied that the electrons are constrained by the attraction of an electron of opposite sign at the centre of the ring: as otherwise their mutual repulsions and the centrifugal forces would produce their dispersion. On the question of loss of energy by radiation from such a system, cf. Ch. XIV. *infra*.

the other in a plane mirror, will present nothing strange to those physicists who regard with equanimity even the hypothesis of the possible existence of both positive and negative matter.

On this view of the constitution of atoms the transit of a material body through the aether does not involve any disturbance in bulk or pushing aside of that medium, unless the body carries an electric charge or is electrically or magnetically polarized.

19. In Maxwell's final presentation of electric theory, in his "Treatise," he deals with displacement but not with anything called electricity**: so that a diagram of molecular polarization is foreign to it. When electric current (recognized electrodynamically) flows from A to B along a wire, the circuit is completed by displacement from B to A through the dielectric: and the notion of charges at A and B is (but only to this limited extent) irrelevant. At the same time there is little doubt that this scheme was the outcome of consideration of the theory of Kelvin and Mossotti, who were the first (in 1845) to extend Poisson's theory of magnetic polarization to dielectrics, of which the electric activity had then just been rediscovered by Faraday: and it seems possible that this notion of electrically polar molecules was dropped by Maxwell because his model of the electrodynamic field did not suggest to him any means of representing the structure of a permanently existing electric pole.

This agnostic attitude as to the nature of electric displacement and electric charge does not however limit the application of his theory on the *electromotive* side, so far as regards bodies at rest; for on any view the most that can be made of conduction in bodies at rest amounts to the direct application of Ohm's law, while the electrodynamics of stationary circuital currents had been already made out by Ampère,

** This statement does not however apply to the memoir 'A Dynamical Theory of the Electro-magnetic Field,' *Phil. Trans.* 1864, in which the theory of discrete electric charges is distinctly indicated; cf. §§ 78, 79. For the demonstration that electrons can have a permanent existence in the rotational aether, cf. Appendix E at the end of this volume.

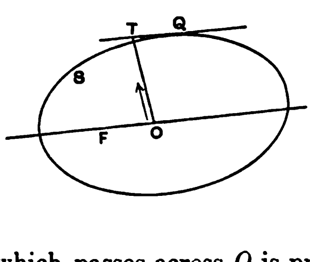
Faraday, and Neumann. But to the case of bodies in motion such a scheme can give no clue, except the first approximation based on the assumption of an equilibrium state of the surrounding field at each instant of the motion. And it can give no account of mechanical or ponderomotive forces, nor therefore of electrostatic phenomena in general, except by the empirical formation in simple cases of a fragment of a mechanical-energy function by taking advantage of the indications of independent observation and experiment.

CHAPTER III

GENERAL KINEMATIC THEORY OF OPTICAL RAYS IN MOVING MEDIA

Specification of a Ray

20. THE relation between the direction of the ray and that in which the radiant waves are travelling is the fundamental conception of optical science. When the material medium transmitting the radiation is at rest in the aether, the ray, or path of the radiant energy, is the same relative to the matter as relative to the aether; and its direction is determined by the wave-surface construction of Huygens, in a manner of which the precise *rationale* is due chiefly to Fresnel. If the point O becomes a centre of radiant activity owing to a train of regular waves advancing on it, the radiation sent on from it travels out into the surrounding space, so that the locus at which a given phase of it has arrived at any instant is a surface S surrounding O, called the wave-surface. Suppose



now that a train of waves is passing across the point O and let the plane F be tangential to a wave-front: draw a parallel plane tangential to the wave-surface on the onward side, the point of contact being Q: then the radiant energy of the portion of this wave-train which passes across O is propagated in the direction OQ. For, if we draw the wave-surface with Q as centre which passes through O, the plane F will be tangential to it at O: hence

the actual radiation sent on from all points of the element of the wave-front F situated at O, which is tangential to that wave-surface, will reach Q in the same phase, as follows from the definition of the wave-surface and the reversibility of the radiation: hence the effects due to all parts of this element of wave-front situated at O will reinforce each other at Q, while those of any other element of the same order of magnitude will obliterate each other owing to differences of phase: thus it is only the portion of the wave-front around O that sends radiation to Q, and the other parts of it may be shut off by screens without altering the effect at Q. It is here tacitly assumed that the medium is homogeneous, so that wave-surfaces of all magnitudes round O are similar, and the ray OQ is therefore a straight line. When there is heterogeneity we must take a wave-surface of very small dimensions, corresponding to a very short time of transit, so that OQ is an element of arc of the ray; the next element of arc starting from Q will now be in an infinitesimally different direction; and thus the ray will be a curved line. The path of a ray between two points P and P' is of course actually explored by placing a source at P and gradually limiting the beam by screens so as not to affect the illumination at P': so long as the screens do not cross the curve PP' constructed as above this will not be affected.

It remains to express these kinematical ideas in analytical form. With a view to this object, we must distinguish, when the medium is of crystalline quality, between the *wave-velocity* and the *ray-velocity* corresponding to any given direction. Thus in the diagram the plane wave-front F is propagated in the direction normal to itself with the wave-velocity appropriate to that direction: but the radiant energy of that wave travels in the direction OQ with the ray-velocity, which is greater than the former in the ratio of OQ to OT, where OT is perpendicular to the tangent plane at Q. When the form of the wave-surface round O is known, the wave-velocities and ray-velocities corresponding to all directions are thereby determined.

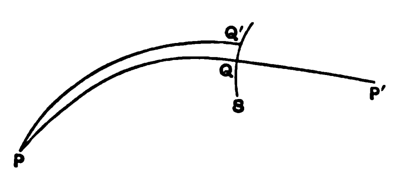
Now the wave-surface S marks the outer boundary of the

region which a radiant disturbance, initiated at O, can affect in a given time. Each ray of the disturbance by following its natural path, with the ray-velocity proper to its direction at each instant, can travel to that bounding surface in the time; but if it is constrained to follow some other path, it cannot get so far in the time. Thus any point on the ray-path OQ is the farthest point on that path that a ray starting from O and guided by any constraint, could possibly reach in the time; and the disturbance actually reaches that point by travelling along the ray itself. That is, the path of a ray from P to P' is that path along which the energy of the disturbance, travelling at each instant with the ray-velocity appropriate to its direction, can pass from P to P' in the least time. This is the generalization, afforded by the theory of undulations, of Fermat's empirical principle*, which asserted that a ray of light travels from one point to another along such path as would make its time of transit least.

This principle remains precisely a principle of least time for paths from P up to all points P' such that the successive wave-fronts between P and P' belonging to a radiant disturbance maintained at P do not develope any singularity along the course of the ray. But when P' lies beyond a place of infinite curvature (cuspidal edge) on the wave-front the principle becomes merely one of stationary time: in certain

imited
ation (cf. p. 36)

cases it may be even a principle of maximum time. A sufficient illustration is afforded by the simple case of rays diverging from P, which after any series of refractions finally emerge into an isotropic medium as straight rays at right angles to a wave-front S. Let the ray from P to P' cross this wave-front at Q: then by definition the time for the ray from P to Q is the same as the time for the ray from P to any consecutive point Q' on this wave-front: in comparing the times for the



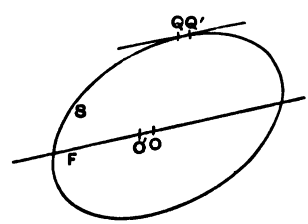
ray PQP' and a consecutive ray $PQ'P'$ we have thus only to

* Cf. Appendix D.

compare the times for the straight segments QP' and $Q'P'$, that is we have only to compare the lengths of these segments. Now clearly from the point P' on a normal QP' to a surface S of double curvatures, $P'Q$ is the line of least length that can be drawn to meet the surface in the neighbourhood of Q, so long as the centres of both the principal curvatures at Q are beyond P'; it is the line of *greatest* length when P' is beyond both these centres; and it is only of *stationary* length, neither maximum nor minimum, when P' is between these centres.

21. Let us consider the form that these principles will assume when the matter across which the radiation is travelling is itself in motion. The radiation is now not reversible, and the demonstration of the law of ray-direction must be expressed differently from the above. This however is easily done.

The time from O to Q is the same as from O to Q', hence is the same as from O' to Q, where OO' is equal and parallel to QQ', each of them being infinitesimal compared with OQ. Hence the disturbances from all points O', near O on the plane wave-front, reach Q at the same time and therefore in common phase, and therefore accumulate, while at a point in any direction other than OQ they would annul each other. Thus the path of a ray is still determined by the principle of stationary time: but the path from P to P' is not the same as the path from P' to P because the velocity of propagation relative to absolute space is altered on reversing the direction of the ray.



In circumstances of moving matter there are moreover two kinds of rays to be distinguished, one of them being the paths of the radiant energy with respect to the particles of the moving matter, the other the absolute paths of the radiant energy in the stagnant aether, or as we may say in space. As radiation is revealed to us wholly by its action on matter, including therein the parts of the eye itself, it is the former

type of rays that is of objective importance. In determining the course of the ray among the elements of moving matter, when it is thus referred to these elements, the principle of stationary time is to be employed using therein a ray velocity which is at each point the velocity of the ray relative to the matter there situated. For that principle ensures that, as the element of matter at Q moves, all the radiant energy arriving at it nearly in the direction of the ray reaches it at each instant in the same phase and thus accumulates: on account of the sameness of time, the path of the relative ray from P to Q is not affected by altering the motions of these terminal points alone.

22. Consider then radiation travelling in a medium of varying density so that the velocity at the point (x, y, z) is V: and let us examine the type of the varying velocity (p, q, r) that may be imparted to the medium, supposed isotropic, without disturbing the forms of the paths *in space* along which the radiant energy travels. We assume, for the present, that the velocity of the ray in space is affected by a fraction k of the velocity of convection of the medium in its direction, the value of k depending on the index of refraction at the place, and being unity for the free aether. When the medium is stationary, the paths of the rays are to be determined by the equation of variations $\delta\int(ds/V)=0$; for the vanishing of this variation ensures that consecutive rays starting in the same phase from one limiting point of the integral shall reach the other limiting point in identical phases, and therefore reinforce each other. When the medium is in motion, the equation of the ray-path in space becomes

$$\delta\int\frac{ds}{V+k\,(lp+mq+nr)}=0,$$

where (l, m, n) is the direction-vector of the element of arc ds, and (p, q, r) is the velocity of the material medium. Thus to the first order of approximation

$$\delta\int\frac{ds}{V}-\delta\int\frac{ds}{V^2}k\,(lp+mq+nr)=0,$$

that is

$$\delta\int V^{-1}ds - \delta\int kV^{-2}(pdx + qdy + rdz) = 0,$$

where V is proportional to μ^{-1}, the reciprocal of the refractive index.

The ray-paths in space cannot remain unaltered unless the second of these terms depends only on the limits of the integral, that is unless $k\mu^2(pdx + qdy + rdz)$ is an exact differential. While if this condition be satisfied, the change in the time of passage of the ray between the two terminal points, arising from the motion of the medium, depends only on the limits of the integral and so is the same for all rays: thus all phenomena of interference between pairs of rays will be unaltered.

We can directly apply this result to a theory of aberration which supposes that the Earth in its orbital motion pushes the free aether in front of it and so sets up a velocity (p, q, r) in it. Our hypothesis will by the principle of relative motion be exact for free aether, k then being unity; thus, taking μ for air to be practically unity, we see that the paths of rays in space will be the same as if the aether were at rest, will therefore be straight, provided $pdx + qdy + rdz$ is an exact differential, that is provided the aethereal motion set up around the Earth is of the differentially irrotational type, in agreement with Sir George Stokes' result.

Thus if the aether around the Earth were set into irrotational motion by the Earth's progress through it, the rays of light from the stars would still travel through space in straight lines: their velocity in space would however be the standard velocity of radiation combined with the velocity of convection of the aether, and so would be affected near the Earth to the order of the ratio of the velocity of the Earth's motion to the velocity of radiation. If now an observer estimated the direction of these rays by looking along two sights situated in free space, or what is practically the same, in open air, his motion and that of the sights would, on the ordinary principles of relative motion, involve an aberrational change in their direction when adjusted to catch the ray, which would be the same as the existing astronomical aberration, except

that its coefficient, being the ratio of the velocity of the Earth to the effective velocity of the ray in space, would, on account of its convection by the moving aether, vary for different parts of the Earth and different times of day by about one part in 10^4, an amount which would not be detected by astronomical observation.

The present object is to make out the best case possible for this type of theory of aberrational effect, which assumes the aether to be set in motion: so we must try to assign a cause for the irrotational quality of motion thus demanded. In free space we have merely to postulate that the aether possesses the properties of the ideal perfect fluid: then by Lagrange's fundamental theorem in fluid motion no convective motion that can be propagated into that medium can be other than irrotational The question then arises how far this explanation will extend to the case in which the aether is entrained by the matter that is moving through it. Attention has already been drawn (§ 10) to Sir George Stokes' considerations which would make the luminiferous property itself prevent the initiation of any rotational motion in the aether. It is in fact not difficult to prove that the energy of strain of a rigid incompressible medium of the type of ordinary matter may be expressed as a volume integral involving only the differential rotation, together with surface integrals extended over boundaries: and it follows that any local beginnings of rotational motion in an aether of elastic-solid type would be immediately carried off and distributed by transverse waves, so that if the rigidity is great enough no trace of rotational motion of the medium in bulk can ever accumulate. In opaque media, however, such waves would not be effectively propagated. The coexistence in the same medium of liquidity for large-scale motions and rigidity for light-waves would on this view be the thing to be explained.

We have been proceeding on the supposition that the Earth's atmosphere moves through the aether without disturbing the motion of the latter, or rather that any disturbance thereby produced does not destroy the differentially irrotational character of its motion. Suppose the observer fixes the direction of the star by an observing telescope instead of by simple

sights, the astronomical law of aberration will still hold provided the motion of the aether in the region inside the telescope retains the same irrotational character as in free space, and not otherwise. But when the tube of the telescope by which the direction of the ray is determined is filled with water instead of air, then if the continuously irrotational character of the aethereal motion were maintained in the water as well as in air, on the lines of Sir George Stokes' dynamical explanation, the course of the ray referred to space, when inside the tube, would not be altered by the motion, and therefore the coefficient of aberration relative to the observer would be reduced in the ratio of the velocity of radiation in water to that in air.

23. This conclusion is contrary to fact. The preservation of irrotational continuity of motion, thus dynamically suggested, must therefore be abandoned: and we are compelled to treat of two interacting media, aether and matter, instead of a simple modified aether. From this new standpoint, in addition to the convection of radiation along with the moving aether, there will have to be a first-order influence on its velocity of propagation in the aether, arising from the relative motion of the matter through it and proportional to its relative velocity. This effect could vanish only if the aether moved along with the matter: whereas if it did so its motion could not be continuous, and also irrotational outside the matter, as in any case it is required to be.

We therefore proceed, for the case of terrestrial rays passing in part through dense media, to develope this wider hypothesis of interacting media. The effective velocity of the rays is now made up of the standard velocity V which would obtain for conditions of rest, diminished by the velocity (P, Q, R) of the space attached to and moving along with the material observing system, increased by the absolute velocity (p, q, r) of the aether itself, and increased by k times the velocity (p', q', r') with which the matter transmitting the radiation is moving through the aether, all measured at the point under consideration and referred to axes fixed relative to the undisturbed distant aether: here k is a constant depending on the nature of the matter, which for an isotropic medium must be scalar.

Thus the velocity, relative to the observing system, of a ray travelling in the direction (l, m, n) is

$$V-(lP+mQ+nR)+(lp+mq+nr)+k(lp'+mq'+nr').$$

Let us first consider the usual case in which all the matter is at rest relative to the observing material system, so that

$$(p'+p,\ q'+q,\ r'+r)=(P,\ Q,\ R);$$

the velocity of the ray is now

$$V-(1-k)(lp'+mq'+nr').$$

Just as before, the condition that the ray-paths relative to the observing system are unaltered to the first order by the common motion of all the matter is that

$$\mu^2(1-k)(p'dx+q'dy+r'dz)$$

shall be the exact differential of a continuous function of position. As (p', q', r') has no necessary connexion with the value of μ, this requires that $\mu^2(1-k)$ shall be a constant A, so that $k=1-A\mu^{-2}$: as moreover k must tend to a null value for very rare material media for which μ is practically unity, we must have A equal to unity so that $k=1-\mu^{-2}$. The condition further requires only that $p'dx+q'dy+r'dz$ shall be an exact differential; so that there is still room for motion of the matter relative to the aether, provided it is differentially irrotational. For example, if we suppose that the aether is stationary, and that the velocity of optical transmission in it is at each point specifically altered after Fresnel's manner, by the fraction $1-\mu^{-2}$ of the velocity of the matter there moving through it, then the ray-paths are unaltered as to form and as to relative phases when the material system to which they are referred and the space attached to it are set into any state of continuous irrotational motion,—the rectilinear motion of the Earth along its orbit furnishing a case in point. This argument shows that the law of Fresnel is on any view required in order to account for Arago's principle.

Let us then proceed, on the basis of Fresnel's value of k thus demonstrated, to the general case in which the material system that transmits the light is in motion with velocity

(p_1, q_1, r_1) relative to the observing system. The velocity of the ray relative to the observing system is now

$$V + (lp_1 + mq_1 + nr_1) - \mu^{-2}(lp' + mq' + nr').$$

Thus the condition that the relative ray-paths are unaltered is that

$$\mu^2(p_1dx + q_1dy + r_1dz) - (p'dx + q'dy + r'dz)$$

should be an exact differential: that is, in addition to the condition already obtained that the absolute motion of the aether itself should be differentially irrotational, we must also have

$$\mu^2 p_1 dx + \mu^2 q_1 dy + \mu^2 r_1 dz$$

the exact differential of a continuous function. In a region of constant index this condition requires that the motion of the matter transmitting the light must relative to the space of the observing system be continuous and irrotational; but at the transition between different substances the tangential components of this motion must be discontinuous so that on the two sides of the interface they are inversely as the squares of the indices of refraction on those sides. These relations are extremely unlikely to be satisfied in actual circumstances.

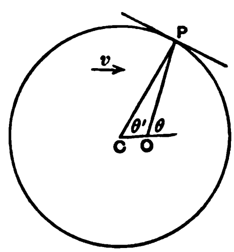
It appears then that there is no optical method of detecting a differentially irrotational flow of the aether superposed on the necessarily existing Fresnel influence of relative motion of the matter on the velocity of propagation in the aether, unless that flow be of cyclic character in the region considered. The Earth's motion might thus, so far as we have yet gone, cause or control such a flow of the surrounding aether, provided however we are willing to admit that bodies of ordinary size at the surface of the Earth are powerless to sensibly deflect it. Such a flow is not required by any optical facts, though Fresnel's effect is so demanded: it is therefore gratuitous to introduce it, especially as it does not in any way simplify our conception of the disturbance imparted to the aether by bodies moving through it; and it will in fact appear that its presence to any sensible extent would introduce excessive complication into a theoretical scheme otherwise simple.

24. The argument above given proves not merely that the principle of Fresnel forms a sufficient basis for the usually received facts of astronomical aberration, but in effect shows that it is necessitated by them. Thus its experimental verification, by Fizeau and by Michelson, was of value rather as a confirmation of the general validity of this line of physical reasoning, than as a special proof of Fresnel's principle itself: if Bradley's law of aberration is granted, in connexion with the observed absence of influence of the Earth's motion on terrestrial ray-paths, that principle follows deductively.

Irrespective then of any experimental evidence relating to the effect of the Earth's motion on interference, diffraction, double refraction, polarization by reflexion, rotatory polarization, or other physical phenomena depending on the dynamical nature of the radiation, considerations of a merely geometrical or kinematical character have restricted the influence of motion of the material medium on the propagation of radiation to the definite relation of Fresnel. It is necessary to find a dynamical basis for that relation, and to show that this basis is in keeping with the relations to the Earth's motion of the other types of phenomena here enumerated, remembering that it is demonstrated only up to the first order of the ratio of the velocity of the material system to that of radiation. Of these other phenomena it may here be noted that interference experiments involve only the relative phases of the rays, and so are included in the above discussion: while the remaining effects above enumerated involve more intimately the dynamics of the waves.

25. In this consideration of ray-paths relative to the moving matter, it has been necessary to include only the first order of small quantities, so that it was unnecessary to distinguish for isotropic media between ray-velocity and wave-velocity. In some discussions which follow, in which the second-order terms are included, it is of course ray-velocity with which we are concerned: and the difference between the two velocities must therefore be determined. When the observing system is moving across aether with uniform translatory velocity v, the velocity of propagation relative to the matter of

a wave-front travelling *in free aether* in a direction inclined to that of v at an angle θ' is $V - v\cos\theta'$: hence the relative wave-surface, being the envelope of simultaneous wave-fronts, is exactly a sphere, say of radius unity, referred to an origin O situated at a distance v/V from its centre C measured in the direction of v. Corresponding to a point P on this sphere, the relative ray-velocity is V multiplied by the vector OP; while the relative wave-velocity is as usual V multiplied by the vector perpendicular from O to the tangent plane at P. Now, θ being the angle between the directions of the ray OP and v, we have



$$OP^2 + CO^2 + 2OP . CO \cos\theta = 1;$$

hence $OP = (1 - CO^2 \sin^2\theta)^{\frac{1}{2}} - CO\cos\theta$; thus the magnitude of the ray-velocity is $(V^2 - v^2 \sin^2\theta)^{\frac{1}{2}} - v\cos\theta$, or up to the second order $V - v\cos\theta - \frac{1}{2}v^2/V^2 . \cancel{\cos^2} \sin^2\theta$. Also, the disturbance relative to the moving matter, that is the ray, is propagated in a direction inclined to the wave-normal at an angle equal, to the first order, to $v/V . \sin\theta$ measured away from the direction of v. These results are on the hypothesis that the propagation relative to the aether itself is isotropic, so that V is independent of direction: otherwise there will also be terms involving interaction between the velocity of convection and the aeolotropic quality.

26. As the axial rotation of the Earth does not come under the restriction above made to an irrotational motion, it follows that our results will not strictly apply as regards the diurnal aberration. In this case there would be an accumulated change of direction in the relative ray-path, in dense matter, for example down a water-telescope, of the order of magnitude of the ratio of the greatest transverse change of the velocity of the matter along the ray to the velocity of radiation: but under no practical circumstances could this be of any importance.

The discussion above given applies to the paths of single rays: it exhibits the conditions under which the time of

passage of the ray from a fixed point A to another fixed point B on it is unaffected to the first order by motion, say of uniform translation, of the matter through which it travels. The condition that A and B should be conjugate foci on the ray is that, for a slight variation of the path of the ray, the time of passage shall be unaffected to the second order of this variation: thus we should be prepared to find that uniform translatory motion of the matter would have a first-order influence on the position along the ray of the focus conjugate to a given one. But there is no means of recognizing an effect of this kind.

It is easy to extend the results here obtained to the case in which deviation is produced by a diffraction grating instead of by reflexion or refraction. The direction of the diffracted ray is determined by the principle that the difference between times of passage from A to B by way of successive physical elements of the grating shall be a period of the light or a multiple of that period: to the first order this difference of times or phases will not be affected by a uniform motion of translation. Thus the principle of Arago and Fresnel extends also to the use of grating spectroscopes. The discussion of the question for more complicated cases of diffraction will be best conducted by aid of the dynamical theory.

Influence of Convection on Radiant Periods: Doppler Effect

27. The disturbance excited in the aether by the uniform motion through it of the radiating body and the Earth cannot affect the change of *relative* period due to approach of the observer and radiant source: for such disturbance will be steady, and therefore the number of vibrations which leave the source in a given time must be the same as the number experienced by the observer, except in so far as his distance from the source has changed in the interval. If we take the aether to be quiescent, the emitted wave-length λ is shortened to $(1 - v/\mathrm{C})\,\lambda$, where v is the velocity of the source resolved in the direction towards the observer and C is the velocity of radiation in vacuum; while the observer, moving towards

the source with velocity whose component is v', picks up these waves with a period τ', which is related to the true period, τ or $(1 - v/\mathrm{c})\,\lambda/\mathrm{c}$, by the equation

$$\tau'(\mathrm{c} + v) = \tau\mathrm{c}.$$

Hence
$$\tau' = \frac{1 - v/\mathrm{c}}{1 + v'/\mathrm{c}}\,\frac{\lambda}{\mathrm{c}};$$

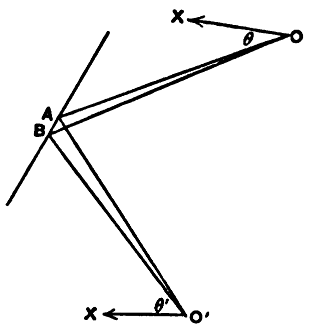
so that the combined motions affect the period in a ratio, which to the first order of small quantities, to which our knowledge is in most respects confined, is equal to $1 - (v + v')/\mathrm{c}$, thus depending only on the relative motion.

The only first-order difference that could arise, according to whether or not the moving body pushed the aether in front of it, would be a difference of wave-length. No such effect would be produced by the source moving the aether in this way, if the receiver is far enough off to be out of the range of the disturbance: but if the velocity v' of approach of the receiver involves a velocity kv' of the aether around it, the wave-length of the radiation relative to the receiver will be altered in the ratio $1 - kv'/\mathrm{c}$. At first sight it appears as if this change of wave-length might be revealed on analysis of the light, for example by the use of a grating which is the type of analysis in which the theoretical conditions are simplest: if that were the case the analysis of light from a terrestrial source, such as a sodium or thallium flame, would show a change of wave-length depending wholly on the relative motion of the matter and the aether around it. But it will be shown generally that uniform relative motion of matter and aether can produce no first-order change whatever either in refractive or diffractive effects: so that no such influence can arise. It seems however worth while to give in brief a special analysis for a grating, in order to illustrate more fully the origin of this absence of effect.

28. The problem under consideration is one in which the observer and all his apparatus are in uniform motion of translation along with the Earth: we have therefore to consider ray-paths and vibration-periods relative to the Earth's motion

through space with velocity v'. Suppose that the aether takes part in this motion to the extent kv': then the relative velocity of radiation in air which is travelling in a direction making an angle θ with the direction OX of the Earth's motion is $V-(v'-kv')\cos\theta$, say $V-v\cos\theta$ where $v=(1-k)v'$.

We have to determine the law of diffraction from a grating under these circumstances. Let ι denote the angle of incidence of the ray OA, say from a sodium or thallium source at O, and ι' the angle of diffraction of the diffracted ray AO', which travels at an angle $\pi-\theta'$ with the direction $O'X$ of the Earth's motion. If we compare the ray OAO' with the ray OBO' which comes from the next reflecting space of the grating, the difference of their times of transit must be a whole number n of complete periods of the radiation, say $n\tau$: this will be true for any point O' on the diffracted ray, up to the first order of the ratio of AB to OA, while for the focus of a curved grating it will be true up to the second order. Thus we have



$$\frac{OA}{V+v\cos\theta}+\frac{AO'}{V-v\cos\theta'}$$
$$=\frac{OB}{V+v\cos(\theta+\delta\theta)}+\frac{BO'}{V-v\cos(\theta'+\delta\theta')}+n\tau$$

where $\theta+\theta'=\iota+\iota'$; or, transposing,

$$\frac{OA}{V+v\cos\theta}-\frac{OB}{V+v\cos(\theta+\delta\theta)}$$
$$=-\frac{O'A}{V-v\cos\theta'}+\frac{O'B}{V-v\cos(\theta'+\delta\theta')}+n\tau,$$

so that to the first order of approximation, for which

$$\delta\theta=AB\cos\iota/OA,\quad \delta\theta'=-AB\cos\iota'/O'A,$$
$$OB=OA+AB\sin\iota\,.\,\delta\theta,$$

we have

$$\frac{-V.AB\sin\iota - OA.v\sin\theta.\delta\theta - AB\sin\iota.V\cos\theta\,\delta\theta}{V^2+2Vv\cos\theta}$$

$$=\frac{-V.AB\sin\iota' - OB.v\sin\theta'.\delta\theta' - AB\sin\iota'.V\cos\theta'\delta\theta'}{V^2-2Vv\cos\theta'}+n\tau,$$

and therefore

$$\sin\iota\left(1-\frac{v}{V}\cos\theta+\frac{v}{V}\cot\iota\sin\theta\right)$$
$$-\sin\iota'\left(1+\frac{v}{V}\cos\theta'-\frac{v}{V}\cot\iota'\sin\theta'\right)=\frac{V}{AB}n\tau,$$

which gives, to the first order in v/V,

$$\sin\iota-\sin\iota'=\frac{V}{AB}n\tau=\frac{n\lambda}{AB}.$$

Thus the law of diffraction does not involve the relative velocity v', which is the required result: it is only changes of period that can affect the law.

For a curved grating the relative motion would affect the focussing, which depends on adjustment of the reduced paths up to the second order: this effect will be of the first order in v/C, but of course wholly beyond the range of detection.

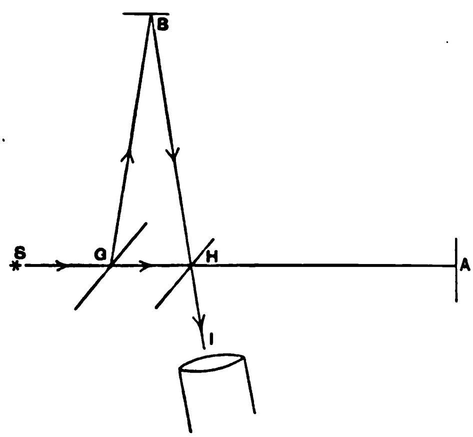
29. It is worthy of remark that, whether the radiation is analysed by a grating or by prisms, the consistency of the results of astronomical measurements of velocities in the line of sight involves incidentally a delicate test of the hypothesis of Arago and Fresnel that uniform motion of the Earth through the aether does not affect the laws of geometrical optics. An uncertainty in the measures of velocity of about one mile per second, which seems at present to be the superior limit in a process that is rapidly improving, would permit discrepancies amounting to only about one second of arc in the optical laws; to recognise this amount in ordinary terrestrial measurement, very exact appliances would be required.

There is one point, however, essential to the theory of measurements of celestial velocities by this means, that has to be settled. How do we make sure that the motion of the source through the aether does not affect the intrinsic periods

of its radiant vibrations? The answer is that the effect, such as it is, must be the same when the translatory motion of the molecules which act as source is reversed, because there is no other velocity or directed quantity connected with the moving molecules such as could enter into combination with their translatory motion: thus the effect on the free periods must depend on the square of the ratio of the translatory velocity to the velocity of radiation, and therefore be far below the order of actual measurements.

Detailed Theory of the Michelson-Morley Interference Experiment

30. The theory of the Michelson and Morley interference experiment*, which is fundamental in this subject, will form an illustration of the principles explained above. A ray of light



from a source S, proceeding in the direction SG of the Earth's motion, is divided by a glass lamina at G inclined to it at an angle $\frac{1}{4}\pi$; the reflected part traces the path $GBHI$ *in space*, being returned by a mirror at B which is parallel to the direction of the Earth's motion, and reaching the lamina again when the point G of it has moved on to H; the transmitted part traces the path $SGAHI$ in space, being returned by a mirror at A at right angles to the direction of the Earth's

* *Phil. Mag.* Dec. 1887.

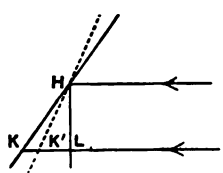
motion. It is still to be proved that the paths in space as thus specified are correct; it has to be shown that the ray SG will be reflected along GB, and that the ray which is returned along AH will be reflected along HI. That being assumed for the present, these two rays, adjusted so as to be both travelling *exactly* along HI, will be the rays that produce interference fringes in an observing telescope directed along IH, when the mirrors A and B are almost exactly equidistant from G; and the circumstances of the interference will be determined by computing the difference of times from G where the rays are separated to H where they are united again. By ray-paths in space we mean for the present ray-paths relative to the aether, which may or may not (so far as we are here concerned) itself be in uniform translatory motion: also GH is the distance the point G of the dividing lamina has moved, relative to the aether, while the reflected portion of the light has traversed the path GBH. If V is the velocity of radiation through aether and v the velocity of the material system relative to the aether, we have the angle GBH equal to $2v/V$, say 2θ; and the angles of incidence and reflexion at B are equal. If the distance of the mirror B from G is l_2, we have $GB = BH = l_2(1+\frac{1}{2}\theta^2)$, and the velocity along each of these lines is V: hence the time over the path GBH is $2l_2/V.(1+\frac{1}{2}\theta^2)$. To find the time over the path GAH, it is easiest to work with velocity of the radiation relative to the material system, otherwise the position of A at the instant of reflexion would have to be found: now if l_1 is the distance of the mirror A from G, the relative velocity from G to A is $V-v$ and from A back to H is $V+v$, hence the time is

$$\frac{l_1}{V-v}+\frac{l_1}{V+v}, \text{ that is, } \frac{2l_1}{V}(1+\theta^2).$$

The coefficients multiplying l_1 and l_2 in the two cases differ by θ^2/V, where θ is equal to v/V: thus if the adjustment to equality of time is made for the system in any position, that adjustment will be disturbed when the whole system is turned through a right angle. The effect of slow steady rotation of the system would thus be a procession of interference bands across the field of the observing telescope, which would reverse four times

(§ 34) in a complete revolution, the number of bands that have crossed between two reversals corresponding to a time-difference of $\theta^2 l/V$ or lv^2/V^3, where l represents l_1 or l_2. But according to the experiments, which have recently been repeated with a refinement that leaves no room for doubt, this effect depending on the square of v/V is entirely absent.

31. It yet remains however to complete the above demonstration by showing that the ray AH is reflected along HI: if that were not so we should have to seek the ray AH' that would be reflected in the direction of HI, and the difference of times up to the focus of the telescope would be affected to the second order in v/V if the inclination of AH' to AH were of the first order. We have thus to determine the law of reflexion, at an advancing mirror, of a ray-system referred to the aether: consider two parallel rays of which one meets the mirror in H; the other would meet it in K if the mirror had not moved forward in the meantime, but really meets it in K' where $KK' : KL = v : V$, v being the velocity of advance of the mirror towards the light: therefore the reflexion is the same as if the mirror were HK' fixed in aether, thus being turned through an angle KHK' or ϵ, where, ι being the true angle of incidence,



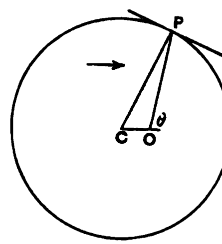
$$\frac{\tan(\iota - \epsilon)}{\tan \iota} = \frac{V - v}{V};$$

so that, ϵ being small,

$$\epsilon = \tfrac{1}{2}\frac{v}{V}\sin 2\iota.$$

In the present case ι is $\frac{1}{4}\pi$; and rotation of the mirror through θ would rotate the reflected ray through 2θ; therefore the ray AH is reflected along HI. In the same way the ray SG is reflected along GB. Moreover it follows from the principle of continuity that practically the same value for the retardation would be obtained by taking any adjacent pair of interfering rays instead of the pair in the diagram. The bands usually observed will naturally correspond to reflexion at the first face of the lamina in each case.

32. *Wave-Velocity and Ray-Velocity in an isotropic Moving Medium.*—In a material medium of index of refraction μ, in uniform translation along the direction of the axis of angular measurement with velocity v, the relative velocity of a train of light-waves travelling in a direction making an angle θ' with the direction v is $\mu^{-1}V - kv\cos\theta'$, where on Fresnel's hypothesis k is equal to μ^{-2}.



As this velocity must be equal to the perpendicular from the origin on the tangent to the wave-surface constructed relative to the moving medium, it follows that this surface is exactly a sphere of radius $\mu^{-1}V$ with its centre C at a distance kv behind the origin O. The ray-velocity relative to the moving system, in any direction OP, is represented by the radius vector OP of the wave-surface: thus if this ray OP makes an angle θ with the direction of v, we have

$$OP^2 + 2kvOP\cos\theta + k^2v^2 = \mu^{-2}V^2,$$

giving

$$OP = -kv\cos\theta + (\mu^{-2}V^2 - k^2v^2\sin^2\theta)^{\frac{1}{2}}$$

$$= \frac{V}{\mu} - kv\cos\theta - \tfrac{1}{2}\frac{\mu k^2v^2}{V}\sin^2\theta,$$

correct up to the second order.

The path of a ray relative to the moving material system would be determined by making the variation of its time of transit between any initial and any final point on the path vanish, using this value of the ray-velocity.

But it is important to remark that the correctness of this second-order term in the relative ray-velocity depends on the assumption that the relative velocity of wave-propagation is as above stated, and thus involves no term depending on $(v/V)^2$. Such a term would be independent of reversal of the direction of v, and therefore could only arise from a constitutive change in the material medium itself, produced by its translation through the aether.

The expression above obtained is correct to the second order for relative ray-velocity in free aether, as in § 25; though, as

has just been seen, there is no reason why it should be correct to that order for a moving material medium, so that any developments derived from it are for ponderable media mainly illustrative. An alteration of the second-order terms in the expression would not however assist towards the explanation of the null result of the Michelson-Morley interference experiment, for the paths of the divided ray are there wholly in air, which is for the present argument practically the same as free aether: thus we are still confined, for the explanation of that result, to the equally reasonable hypothesis of a second-order change in the linear dimensions of the solid material system of the experiment, arising from its motion through the aether (§ 112).

33. *General Analysis of Interference in Moving Media.*—The problem of optical interference in moving material media may be treated in a quite general manner. It has already been seen that, on Fresnel's hypothesis, the relative paths of the rays in a uniformly moving material medium of varying density are the same as if the system were at rest, up to the first order of small quantities inclusive. Further it has been shown that the relative velocity of the *ray* which travels at an angle θ with the direction of the uniform translational velocity v of the system is $\mu^{-1}V(1 - k\epsilon\cos\theta - \frac{1}{2}k^2\epsilon^2\sin^2\theta)$, where ϵ is equal to $\mu v/V$, while on Fresnel's hypothesis k is equal to μ^{-2}. Thus if $\delta s'$ denote an element of any ray-path of continuous curvature, relative to the system in motion, and δs the corresponding element of the ray-path if the system were at rest, the time of passage of the ray for the moving system is

$$\int \frac{\mu ds'}{V}(1 - k\epsilon\cos\theta - \tfrac{1}{2}k^2\epsilon^2\sin^2\theta)^{-1},$$

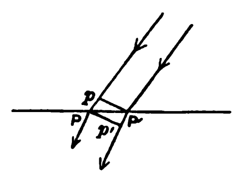
which is equal up to the second order inclusive to

$$\int \frac{\mu ds}{V}(1 - k\epsilon\cos\theta - \tfrac{1}{2}k^2\epsilon^2\sin^2\theta)^{-1} + \int \frac{\mu}{V}(ds' - ds).$$

Of this expression the second term is the difference of the times of transit of a ray over the path s' and over the natural path s, when the medium is at rest. Now Fermat's principle of least time shows that if these paths differed in position up

to the first order, the times of transit would differ only by the second order: but actually the paths differ in position only up to the second order, hence this term is negligible up to that order. We can therefore calculate the time of passage of a ray relative to the moving material system, correctly up to the second order, by assigning to it the path s that would actually belong to it when the material medium is at rest.

This proposition holds good however abrupt the transition of density may be at certain surface-loci: hence it really includes as limiting cases those in which the continuity of curvature of the ray is disturbed by a finite number of reflexions or refractions. It is however easy to see independently that these cases do not introduce any disturbance into the result. For consider a refraction as in the diagram, $P'p'$ being the actual ray relative to the moving medium, and pP that which it is proposed to substitute for it, namely the corresponding ray in the medium at rest. We have seen that the length PP' is of the second order of small quantities. The special effect of the refraction on this substitution is to add the element of arc Pp to the integral for the time of passage, and to take away the element of arc $P'p'$: these are both of the second order, thus to that order the change produced is to add to the integral $\mu_1 . Pp$, and to subtract $\mu_2 . P'p'$, where μ_1 and μ_2 are the refractive indices above and below the interface: but these terms are equal, each representing the time of passage from the wave-front $P'p$ to the wave-front Pp': hence there is no change in time here introduced, up to the second order.



34. Now when the medium is at rest the paths of the interfering rays in the Michelson-Morley arrangement are as in the diagram, each of the rays being reflected straight back to the mirror which originally divided them. Thus if the velocity v of the material system makes an angle θ with the direction of the incident light, the times of transit of the two rays relative to the moving system are, up to the second order,

$$\mu l_1 / V (1 - k\epsilon \cos\theta - \tfrac{1}{2}k^2\epsilon^2 \sin^2\theta)$$
$$+ \mu l_1 / V (1 + k\epsilon \cos\theta - \tfrac{1}{2}k^2\epsilon^2 \sin^2\theta)$$

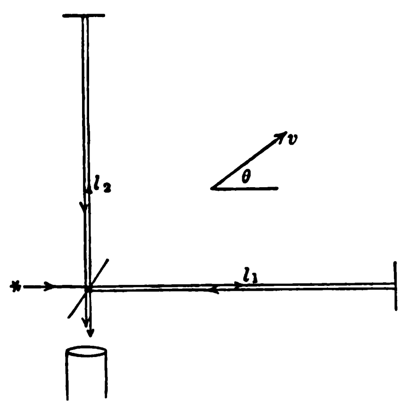
and

$$\mu l_2/V\,(1 - k\epsilon \sin\theta - \tfrac{1}{2}k^2\epsilon^2 \cos^2\theta)$$
$$+\ \mu l_2/V\,(1 + k\epsilon \sin\theta - \tfrac{1}{2}k^2\epsilon^2 \cos^2\theta);$$

these are equal to

$$\frac{2\mu l_1}{V}\,(1 + k^2\epsilon^2 - \tfrac{1}{2}k^2\epsilon^2 \sin^2\theta)$$

and $$\frac{2\mu l_2}{V}\,(1 + k^2\epsilon^2 - \tfrac{1}{2}k^2\epsilon^2 \cos^2\theta) \text{ respectively.}$$



The difference of times of transit is thus

$$2\mu\frac{l_1 - l_2}{V} + \mu\frac{l}{V}k^2\epsilon^2 \cos 2\theta,$$

since in the second-order terms we may write l for l_1 or l_2. Thus, as the apparatus revolves, the fringes pass backwards and forwards across the field of view of the observing telescope with a simple harmonic oscillation, moving in the same direction during a quadrant of the revolution. The total change of phase between the two positions, in which the light is incident along and at right angles to the Earth's motion, is

$$\frac{2l}{V} \cdot \frac{v^2}{\mu V^2}.$$

The effect of inserting a tube of water in one of the arms so that part of the path is in air and part is in water, may be

easily estimated in this way; the result will still of course be of the second order. Thus if the path l_1 consists of a part l_1' in air and a part l_1'' in water of index μ, while the path l_2 is all in air, the difference of times of transit would come out as

$$\text{constant} - \frac{1}{2}(l_2 + l_1' + \mu k^2 l_1'')\frac{\epsilon^2}{V}\cos 2\theta,$$

where approximately

$$l_2 - l_1' = \mu l_1''.$$

But it is to be borne in mind, as above explained, that this result neglects the second-order effect on the velocity of radiation in the material medium due to constitutive change in it arising from its motion through the aether, and also the effect on the linear dimensions of the material system arising from the same cause. When these effects are included, the result will probably, on any view, be quite different: according to the general molecular theory to be explained later, it will always be null.

CHAPTER IV

THE PROBLEM OF OPTICAL CONVECTION: INDICATIONS TOWARDS A DYNAMICAL THEORY

35. CONSIDER in the first place the propagation of waves—or indeed the course of any kind of disturbance—in a single self-contained medium, with a view to determining the effect on it of a velocity of uniform translation imparted to the medium. The principle of relative motion supplies the solution. Impart to the whole system a velocity equal and opposite to that of the medium; and, because this uniform velocity introduces no new kinetic reactions, the phenomena of the relative motion will pursue the same course as before, but they will now be relative to the medium at rest instead of in motion. In all such cases, therefore, the disturbances in the medium are simply carried on along with the medium itself, with its full velocity of translation, and in other respects pursue their course unaltered.

For example, the velocity of translation of the air through which sound is propagated is added (in algebraic sense) at each instant to the velocity of the sound itself. In the same way, if, adopting the view discussed by Sir G. Stokes, we considered the surrounding free aether as disturbed by the Earth's motion, its velocity at each point would have to be added on to the intrinsic velocity of radiation through it. This is true whether the motion of the medium is uniform or not: when however it is not uniform a simple wave will no longer travel as a simple wave, and to that extent the meaning of the term velocity of the wave is indefinite.

In expressing the velocity of a particle of the moving system relative to coordinate axes travelling with the system

$$d/dt + vd/dx$$

must replace d/dt, where v is the velocity of the system and is taken to be parallel to the axis of x. We may illustrate by the simple case of the propagation of waves along a stretched cord which is carried through two fixed eyelets, and runs through them with a uniform velocity of translation v. Considering transverse waves, if η denote the transverse displacement at a distance x along the tight cord, the transverse velocity is $(d/dt + vd/dx)\,\eta$, and the transverse acceleration of this element of the cord is $(d/dt + vd/dx)^2\,\eta$. The tension T in the cord is uniform because v is uniform; in fact any slight difference of tension, however initiated, is smoothed out by longitudinal waves which are assumed to travel very much faster than the transverse waves under consideration. Thus the restoring force is as usual $T\frac{d}{dx}\frac{d\eta}{dx}\delta x$ per length δx: and the equation of propagation is

$$\rho\delta x\left(\frac{d}{dt} + v\frac{d}{dx}\right)^2 \eta = \delta x T\frac{d}{dx}\frac{d\eta}{dx}$$

or

$$\left(\frac{d}{dt} + v\frac{d}{dx}\right)^2 \eta = C^2\frac{d^2\eta}{dx^2},$$

where C^2, equal to T/ρ, is the normal velocity of propagation. Assuming a solution in the form of the simple wave-train

$$\eta = \eta_0 \exp \iota\frac{2\pi}{\lambda}(x - Vt),$$

this gives the relation $(V - v)^2 = C^2$, so that $V = C + v$ exactly, as was to be anticipated.

For purely longitudinal waves, of displacement ξ, the equation of propagation is

$$\rho\left(\frac{d}{dt} + v\frac{d}{dx}\right)^2 \xi = \frac{d}{dx}\left(E\frac{d\xi}{dx}\right),$$

where E is the longitudinal elasticity, and $(E/\rho)^{\frac{1}{2}}$ is the undisturbed velocity of propagation. As in the previous case,

this velocity is increased by the velocity of translation of the cord measured positive towards the direction of propagation of the waves.

36. Let us proceed now to the propagation of electric waves across a dielectric medium, which is moving with uniform velocity v parallel to the axis of x. If we follow Maxwell's scheme of equations, and his notation, we have

$$P=-\frac{dF}{dt}-\frac{d\Psi}{dx},\ Q=-vc-\frac{dG}{dt}-\frac{d\Psi}{dy},\ R=vb-\frac{dH}{dt}-\frac{d\Psi}{dz}, \ldots\text{(i)}$$

where $(a, b, c)=\left(\frac{dH}{dy}-\frac{dG}{dz},\ \frac{dF}{dz}-\frac{dH}{dx},\ \frac{dG}{dx}-\frac{dF}{dy}\right)$;

also $4\pi(u, v, w)=\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz},\ \frac{d\alpha}{dz}-\frac{d\gamma}{dx},\ \frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)$

and $(a, b, c)=\mu(\alpha, \beta, \gamma)$;

yielding $\nabla^2(F, G, H)=-4\pi\mu(u, v, w)$* ....(ii).

These equations are satisfied by the propagation of a train of transverse waves along the axis of x, in which P and a and u and F are null, while Ψ is a function of y, z: thus for such a wave-train they give

$$Q=-\left(\frac{d}{dt}+v\frac{d}{dx}\right)G-\frac{d\Psi}{dy}$$

$$R=-\left(\frac{d}{dt}+v\frac{d}{dx}\right)H-\frac{d\Psi}{dz}.$$

As $\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}$ is null in accordance with (ii), any theory that makes

$$\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}$$

null will also make $\nabla^2\Psi$ null, that is will make Ψ merely the static potential of the electric charges in the field. Then we shall have

$$\nabla^2 Q=4\pi\mu\left(\frac{d}{dt}+v\frac{d}{dx}\right)v,$$

$$\nabla^2 R=4\pi\mu\left(\frac{d}{dt}+v\frac{d}{dx}\right)w;$$

* Cf. § 55, and end of Appendix A.

and the remainder of the analysis will depend on the relation that is adopted between the dielectric current (u, v, w) in the moving medium and the electric force (P, Q, R).

(1) If we were to *assume* the ordinary relation for a medium at rest, namely

$$(u, v, w) = \frac{K}{4\pi C^2}\frac{d}{dt}(P, Q, R),$$

the above condition of nullity of

$$\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz}$$

would be satisfied, and the equations of propagation would be

$$\nabla^2 (Q, R) = \frac{K\mu}{C^2}\frac{d}{dt}\left(\frac{d}{dt} + v\frac{d}{dx}\right)(Q, R).$$

On introducing the type of a simple wave-train

$$(Q, R) = (Q_0, R_0)\exp\iota\frac{2\pi}{\lambda}(x - Vt),$$

they would give $$V(V - v) = \frac{C^2}{K\mu}$$

so that $$V = \frac{C}{(K\mu)^{\frac{1}{2}}} + \tfrac{1}{2}v + \tfrac{1}{8}\frac{(K\mu)^{\frac{1}{2}}}{C}v^2, \text{ approximately;}$$

thus the velocity of the wave-train would be increased, to a first approximation, by half the velocity of translation of the medium.

(2) If we assumed that the whole of the system, aether and matter, that is polarized or otherwise affected by the electric force, moves together, with the uniform velocity v, and that the change of its actual polarization constitutes the dielectric current, we should have

$$(u, v, w) = \frac{K}{4\pi C^2}\left(\frac{d}{dt} + v\frac{d}{dx}\right)(P, Q, R);$$

and the equation of propagation would be

$$\nabla^2 (Q, R) = \frac{K\mu}{C^2}\left(\frac{d}{dt} + v\frac{d}{dx}\right)^2 (Q, R).$$

The waves would then partake of the whole of the velocity of translation of the medium.

This rather than the previous result in (1) is what we should expect on the dynamical principle of relative motion, when the whole system transmitting the waves is involved in the translatory motion. We therefore conclude that on such an aspect of Maxwell's theory—one namely which considers everything to partake in the motion—the present relation between the dielectric current and the electric force in the moving medium would be the right one.

But neither of these results is in agreement with the facts, which in very rare media such as gases make the influence on the velocity of the waves extremely small. So we conclude that in a rare medium the main part of the electric flux, which is then the part connected with the aether itself, is not convected at all with the moving material system. We are therefore led to another hypothesis,

(3) which divides the total dielectric current into an aethereal part $\frac{1}{4\pi C^2}\frac{d}{dt}(P, Q, R)$, which is not convected presumably because the aether does not participate at all in the motion of the matter, and another part depending on material polarization which is convected to the full extent and therefore is of the form $\frac{K-1}{4\pi C^2}\left(\frac{d}{dt}+v\frac{d}{dx}\right)(P, Q, R)$. Thus for the total current, referred to axes at rest, we would have

$$(u, v, w)=\frac{1}{4\pi C^2}\left\{\frac{d}{dt}+(K-1)\left(\frac{d}{dt}+v\frac{d}{dx}\right)\right\}(P, Q, R),$$

leading to equations of propagation

$$\nabla^2(Q, R)=\frac{\mu}{C^2}\frac{d}{dt}\left\{\frac{d}{dt}+(K-1)\left(\frac{d}{dt}+v\frac{d}{dx}\right)\right\}(Q, R).$$

The equation for the velocity is now

$$V\{V-(1-K^{-1})v\}=\cancel{\frac{K\mu}{C^2}}, \frac{C^2}{K\mu},$$

so that $$V=\cancel{\frac{(K\mu)^{\frac{1}{2}}}{C}}+\tfrac{1}{2}(1-K^{-1})v, \text{ approximately.}$$

$$\frac{C}{(K\mu)^{1/2}}$$

The change of velocity of the waves is now just half of that given by the formula of Fresnel, which has been fully verified by experiment. Thus we are impelled a stage further, and led to inquire why the displacement current in the stagnant aether should have to do at all with the electric force (P, Q, R), which involves in its constitution the velocity v of the matter carrying the electric charges on which alone electric force operates. If we assume that this aethereal electric current is excited by the same cause as produces it when there is no matter present, or when the matter is at rest, namely by what we may call the aethereal force (P', Q', R'), connected with the electric force by the relation

$$(P', Q', R') = (P, Q + vc, R - vb),$$

we shall have to combine an aethereal displacement current

$$\frac{d}{dt}(f, g, h), \text{ or } \frac{1}{4\pi C^2}\frac{d}{dt}(P', Q', R'),$$

and a material polarization current

$$\left(\frac{d}{dt} + v\frac{d}{dx}\right)(f', g', h'), \text{ where } (f', g', h') = \frac{K-1}{4\pi C^2}(P, Q, R),$$

in order to obtain the total dielectric current, which will thus be

$$\frac{1}{4\pi C^2}\left\{\frac{d}{dt}(P', Q', R') + (K-1)\left(\frac{d}{dt} + v\frac{d}{dx}\right)(P, Q, R)\right\}.$$

Now putting Ψ null for purely transverse waves, which will be found to cause no discrepancy, we have

$$(Q', R') = -\frac{d}{dt}(G, H);$$

and, keeping now to G, H as more convenient independent variables, we derive

$$\nabla^2 G = \frac{\mu}{C^2}\left\{\frac{d^2G}{dt^2} + (K-1)\left(\frac{d}{dt} + v\frac{d}{dx}\right)^2 G\right\},$$

with the similar equation for H. These equations give for the velocity of propagation

$$V^2 + (K-1)(V-v)^2 = \frac{C^2}{\mu},$$

or $$KV^2 - 2(K-1)vV = \frac{C^2}{\mu} - (K-1)v^2,$$

that is $$V = (1 - K^{-1})v + \left(\frac{C^2}{K\mu} - \frac{K-1}{K^2}v^2\right)^{\frac{1}{2}}$$

$$= \frac{C}{(K\mu)^{\frac{1}{2}}} + (1 - K^{-1})v - \left(\frac{\mu}{K}\right)^{\frac{1}{2}}(1 - K^{-1})\frac{v^2}{2C}, \text{ approximately,}$$

agreeing to the first order of small quantities with Fresnel's formula.

The principles to which the above cursory preliminary sketch has pointed, form the basis of the definite dynamical theory of the electrical and optical relations of moving material media, which will be worked out in detail in the following pages.

37. If we determined to avoid the introduction of the auxiliary vector potential (F, G, H), this argument would have to be expressed as follows. Let (f, g, h) denote the aethereal part and (f', g', h') the material part of the total electric displacement of Maxwell; the circuital electrodynamic relations, for the moving material medium will, when referred to axes at rest in the quiescent aether, be of types

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi\left(\frac{df}{dt} + \frac{\delta f'}{dt}\right),$$

$$\frac{dR}{dy} - \frac{dQ}{dz} = -\frac{\delta a}{dt}, \text{ where } \frac{\delta}{dt} = \frac{d}{dt} + v\frac{d}{dx}; \text{ and}$$

ll. 1 Corr. ~~where (P', Q', R') represents $4\pi C^2(f, g, h)$, and where d'/dt when it operates on (f, g, h) is the same as d/dt, but when it operates on (f', g', h') is the same as δ/dt or $d/dt + vd/dx$.~~

Further $(f+f', g+g', h+h') = K(f, g, h)$, just as when the material medium is at rest: for its motion cannot alter the value of K to the first order of small quantities.

This scheme of equations forms a sufficient basis for the theory, without any direct assumption as to the relation between (f', g', h') and (P', Q', R'): the relation previously given

$$(f', g', h') = \frac{K-1}{4\pi C^2}(P', Q' - vc, R' + vb)$$

is in fact implicitly involved in this scheme. Thus the distinction that is necessary in moving media between the electric

force and the aethereal force is involved in the circuital relations as here expressed.

The first of this system of equations gives, in the *general* problem of dielectric propagation, equations of type

$$\frac{d}{dy}\frac{d'\gamma}{dt} - \frac{d}{dz}\frac{d'\beta}{dt} = 4\pi\left(\frac{d^2f}{dt^2} + \frac{\delta^2 f'}{dt^2}\right)^{**};$$

whence by substitution from the second we obtain for a homogeneous medium the three equations of type

$$\nabla^2 P' = 4\pi\mu\left(\frac{d^2f}{dt^2} + \frac{\delta^2 f'}{dt^2}\right),$$

that is

$$\frac{C^2}{\mu}\nabla^2 f = \frac{d^2f}{dt^2} + (K-1)\left(\frac{d}{dt} + v\frac{d}{dx}\right)^2 f,$$

with the similar equations for (α, β, γ). These lead to Fresnel's expression for the convection-effect, as before.

38. It will be shown later (Ch. X), more generally, in connexion with molecular theory, that if any system of electrons exist at rest in the aether with ideal rigid connexions between them, and its state is compared with that of the same rigidly connected system of electrons in motion with uniform translatory velocity v through the aether, then, when the square of v/C is neglected, (i) the forces which act on the individual electrons are the same in the two cases, (ii) a correspondence can be established between aethereal disturbances propagated across the system from one group of electrons to another in the two cases, so that though electric and magnetic displacements do not correspond yet relative wave-fronts do, and a place where there is no disturbance in the one system corresponds to a place where there is no disturbance in the other. Now all, or almost all, exact electrical and optical measurements are made by null methods: that is, a moveable piece of apparatus is introduced into the system and so becomes part of it, and observation is made of its position when a certain kind of disturbance is just obliterated. All such experimental determinations will therefore be the same, up to the first power

** This simply means that the rate of change of the integral of magnetic force round a small fixed circuit is equal to 4π times the rate of change of the current through its aperture.

of v/C, in the fixed and the moving system: there will be no possibility, except it may be as regards the second order of v/C, of deciding whether the system is at rest or in uniform motion through the aether, by means of phenomena which occur wholly within the system itself.

As an illustration, consider the rotation of the plane of polarization of light in passing through quartz. On formulating a direct analytical theory of the effect, and transforming the equations to axes moving with the matter, assuming that the value of the rotatory coefficient of the matter is not altered by the motion, we should obtain, according to Lorentz's analysis, a first-order effect arising from the motion of the Earth through space, which is greater than would escape detection. Yet consider the system formed of polarizer, quartz plate, analyzer: if it is so arranged as to prevent the incident luminous disturbance from getting through when the Earth is at rest, it should, by the above general result, remain thus arranged, correctly up to the first power of v/C, when the motion of the Earth intervenes: and thus change of direction of the Earth's motion should not have any first-order effect on the adjustment. This is in keeping with Mascart's experimental result, of which the validity up to the first order of v/C admits of little doubt. According however to Lorentz's analysis there ought to be a first-order effect when the optical rotatory coefficient is supposed unaltered. If that were so we should, in the light of the general principle, have to compensate this effect by assuming an alteration in the rotatory coefficient arising from the motion. It will be seen (§ 92) that *à priori* there is no formal objection to the existence of a new constituent of the rotatory power, arising from this cause, which would be of the first order: but this new term would be related to a directed quantity, namely the velocity of the motion, and therefore it would be of the type of magnetic rotation, and thus could not, except accidentally in a particular case, compensate an effect that is structural, as Lorentz's term is. It will appear (Ch. XIII.) that the discrepancy is cleared up by the existence of error in Lorentz's analysis: and that Mascart's result indicates that there is in fact no first-order modification

at all in rotatory optical quality arising from convection of the material medium.

A very large number of optical phenomena have been examined by various experimenters with a view to detecting an influence on them of the Earth's velocity of translation. The only such influence that has been announced is that found by Fizeau on the displacement of the plane of polarization of light, produced by transmission through a pile of glass plates: according to Fizeau's own view the experiment was uncertain owing to the numerous disturbing causes that had to be guarded against; and this doubt as to the feasibility of the observation has been fully shared by Maxwell and most other authorities who have considered the matter.

The only cases in which a first-order effect of the motion of the medium is to be anticipated theoretically are those in which the optical or other disturbance that is examined comes from outside the uniformly moving system which includes the observer. The known instances, which are fully covered and explained by the first-order theory just mentioned, are the Doppler effect of change of optical period arising from the relative motion of the source and the observer, and the astronomical aberration of light.

39. An interference experiment on the difference of the times of propagation round two cyclic paths, originally suggested by Maxwell, has been carried out by Michelson and Morley (§ 30) with the negative result anticipated on all theories as far as the first order is concerned. It occurred to them that by aid of very high refinement in the experimental arrangements the terms of the second order, which can effect a discrimination, might be successfully examined: the result of the experiments, which have recently been repeated with still further refinement and delicacy, has been to make it reasonably certain that the terms of the second order also vanish, that in fact the time of propagation is independent of the Earth's motion not merely to a first approximation but to a higher order. The theory as hitherto developed in terms of the physical constants of the material media has nothing to say

to this result, because it is not in a position to assign the second-order changes that the Earth's motion produces in the physical constants of material media and in the instrumental arrangements. But the purely negative result is in itself an important clue towards an extension of theory to the second order, in which we must necessarily deal with the molecular structure of the medium.

Hitherto in treating in this molecular manner of the change of distribution of a free electric charge owing to the Earth's motion, the electrons of the charge have been supposed to be rigidly fixed in the positions they would occupy when the conductors are at rest, and the additional forces to which the motion would subject them are calculated. It is found that these forces vanish up to the first order: so that to that order no change in the distribution will result. But in proceeding to higher orders we must deal with the problem as one of pure aether in which each electron is a singular point. It is found, on transformation to axes of coordinates (x', y', z') moving with the electrons, that corresponding to each resting configuration expressed by functions of x, y, z and t there is a moving one, expressed by the same functions of $\epsilon^{\frac{1}{2}}x', y', z'$ and t', where $t' = t - vx/C^2$ and $\epsilon = 1 + v^2/C^2$, which has the same electrons in corresponding positions, and also the wave-fronts of radiation traversing it in corresponding positions. The inference is made that the change from x' to $\epsilon^{\frac{1}{2}}x'$ is a real shrinkage of the material system, and that after this has happened the courses of all the phenomena above mentioned are identical in the two systems up to the second order. This inference rests on the hypothesis that an electron is nothing more than a point-singularity or pole in the electrodynamic and optical aether, and that the atoms of matter are constituted of aggregations of such poles. Should it turn out that the atoms have also inertia and mutual forces of other kinds than this view involves, which however must arise from another entirely different set of properties of the aether to which no clue has yet appeared, the argument would lose its validity even were this extraneous inertia proportional to the intrinsic electric charge. It is inferred that Michelson's negative result supports the widely

held conclusion that the main part of the actions, chemical and other, between molecules and between the constituent parts of molecules, is of electrodynamic type: that if gravitation and possibly some actions of cohesion stand outside that type, their importance, considered as regards molecular relations, is slight compared with that of the electric actions.

Can the convection of electrically charged bodies along with the Earth affect a magnetometer?

40. Under no circumstances can the motion through space, with uniform velocity of translation, in which a system of charged conductors participates along with the Earth, produce any magnetic force in a region shielded by a conducting screen from outside electrostatic influence. For consider the influence of a single point-charge q of the system, which is moving with uniform velocity v parallel to the axis of x: the magnetic force due to it at a point whose distance r makes an angle θ with v is, to the first order, $qvr^{-2}\sin\theta$ tending around the direction of motion of the charge. Thus, taking the point as origin, it is made up of components $qvr^{-3}z$ parallel to the axis of y and $-qvr^{-3}y$ parallel to the axis of z; while for any system of such charges the effect is obtained by summation. Now at a point inside a conductor in a steady state, situated in a magnetic field (a_0, b_0, c_0), the total electric force, which is thus equal to

$$(C^2\Sigma qr^{-3}x,\ C^2\Sigma qr^{-3}y - vc_0,\ C^2\Sigma qr^{-3}z + vb_0),$$

must vanish. Hence the magnetic force due to translation of the charged bodies with uniform velocity vanishes to the first order of v/C, compared with that of the field, throughout any space shielded off from the charges by a conducting body; the reason being that a countervailing charge is induced on the surface of this conducting screen.

This accounts for the negative result of Röntgen's experiments, in which he tested whether the convection of a charged body along with the Earth affected the orientation of a compass needle in its neighbourhood. The charged body here induces a countervailing electric charge on the electric screen protecting the compass needle, or on the surface of the needle

itself, such that whatever be the direction of the Earth's velocity through the aether, the actions of these two charges on the magnetic elements which make up the body of the needle exactly neutralize each other. At first sight it might appear that when the countervailing charge is on the surface of the needle itself, it could exert no resultant influence, because the action of the needle on this charge would be equal and opposite to that of the charge on the needle. But it will appear (§ 41) that the electric force arising from the convection of the magnetic needle is derived from a potential $-vF$, and therefore is also countervailed by an electric distribution in the needle and on its surface (cf. § 67) which prevents it from affecting the superficial charge, and is itself not affected because there is no electric force. The effect of the motion of a permanent magnet on its own constitution must be of the second order of small quantities and so will not enter here, because that effect is not altered by a reversal of the velocity.

A more delicate question, though not a practical one, arises if we imagine the permanent magnet to be made of dielectric material, and not screened off electrostatically from the moving charges. In this case, as before, the magnetic force at any point due to the convection of the electric charge with uniform velocity v parallel to the axis of x is $c^{-2}(0, -vR, vQ)$*; where (P, Q, R) is the electric force due to the charges, which is not now compensated by the shielding of an induced superficial charge. Thus there would appear to be in this case a real magnetic force throughout the magnet arising from the convection of the charges; so that, if there could be such a dielectric permanent magnet (and if the direct electrostatic action could be experimentally allowed for), a convection effect of the kind here considered might be expected.

41. In the same way a converse influence of the uniform translatory motion of the magnets, through the aether along with the Earth, on the electric force might at first sight be

* More generally a system of electrons or charged bodies whose electric field is (P, Q, R) will, when moving with steady uniform velocity (p, q, r), produce a magnetic field $c^{-2}(qR - rQ,\ rP - pR,\ pQ - qP)$.

anticipated. Generally the kinetic part of the electric force (i.e. of the force which acts on the electrons of material bodies) is by § 59 $(\gamma\dot{y} - \beta\dot{z} - \dot{F}, \alpha\dot{z} - \gamma\dot{x} - \dot{G}, \beta\dot{x} - \alpha\dot{y} - \dot{H})$; thus when as before $(\dot{x}, \dot{y}, \dot{z}) = (v, 0, 0)$ and the system is in a steady state of translation, so that $\dot{F} + vdF/dx$ is null, there is a direct change in the electric force of amount $(vdF/dx, vdF/dy, vdF/dz)$ arising from the motion, assuming as is natural that the value of (F, G, H) at any point is not sensibly affected thereby. This additional term in the electric force is derived from a potential and so will not disturb electric currents. It will not even tend to alter the electric distributions on conductors in the neighbourhood: for it can be represented as arising from an ideal electric distribution within the magnets and on their surfaces (§ 67), so that if these magnets are conducting bodies, what would happen would be that actual electric distributions would be induced throughout their volumes and over their surfaces which would neutralize this part of the electric force for their interiors and at the same time shield them off from the surrounding space: thus here again no effect would arise*.

* The existence of an effect, of this kind also, is suggested by Wien, *Wied. Ann.*, July 1898.

SECTION II

CHAPTER V

ON METHOD IN GENERAL PHYSICAL THEORY

On the Scientific Use of Hypotheses

42. THE cultivation of *à priori* physical theories of purely abstract type is not merely an affair of philosophical speculation. Their practical necessity for scientific progress, as also the amount of uncertainty that is inherent in them, may conveniently be illustrated by a review of some chapters of the scientific history of our present subject.

The master idea of Roemer that the delay in the observed eclipses of Jupiter's satellites, when the planet is in the part of its orbit furthest removed from the Earth, is due to the interval of time required by light to transmit the event across the intervening space to the terrestrial observer, was at the time when it was enunciated an effort of pure scientific imagination, for which the evidence lay solely in the intellectual simplicity of the explanation which it afforded**. This evidence was many years afterwards very materially strengthened by Bradley's cardinal discovery of the astronomical aberration of light: for

** The idea that light may travel with finite velocity seems to have originated, so far as regards modern physics, with Galileo, who had an intention of submitting the subject to experiment. According to Descartes' ideas, light was a sort of impulsive pressure which spread out instantaneously throughout space, in favour of which view he claimed that if the velocity were finite, eclipses would be seen at an interval after their real times of occurrence; this is precisely the principle that guided Roemer to his estimate of the velocity of light.

it was recognized that if light consists of corpuscles moving towards the observer, with a definite speed for each medium, then the apparent direction from which they come must be affected by motion of the observer exactly as Bradley's law requires. The corroboration thus obtained for the hypothesis of the finite velocity of light was powerful and legitimate, and the ideas involved in that hypothesis had much to do with the evolution of Bradley's great discovery, notwithstanding that the physical scheme involved in his use of it, that namely of the corpuscular theory of light, was not merely imperfect but positively erroneous. Had there been independent means at that time of arriving at a tolerable estimate of the Sun's distance, this train of physical deduction would have had something very substantial to confirm the net of pure hypotheses on which it was supported: for it would then have been possible to verify the identity of two values of the velocity of light derived from entirely independent sources. In the cognate case of the electrodynamic theory of radiation, a numerical corroboration of this kind was, for a considerable series of years, the only experimental evidence that was forthcoming for a scheme which originated with Maxwell as a train of purely hypothetical deduction, and was on that ground refused acceptance by weighty authorities. Our present object, however, is to notice that as matters stood at the beginning of the present century, there was in existence a compact and reasoned theory of the finite propagation of light, constructed wholly on the corpuscular view: that this theory, though actually on wrong lines and not merely incomplete, was yet a useful hypothesis in its day, in that it gave a constitution to radiation that in certain ways was so analogous to its actual constitution that it served as a basis for great practical advances in astronomical and optical science. We have thus an illustration of the fact that a hypothetical scheme may serve as a useful instrument for the progress of Natural Philosophy, notwithstanding that more minute scrutiny may subsequently prove it to be not merely imperfect but quite on a wrong track. There are in fact two ways in which such a hypothesis may work: it may lead readily to deductions which are really

logically involved in the facts that suggested the hypothesis, and which will therefore be verified by observation and lead on to deeper knowledge: but on the other hand it may lead to results which intrinsically depend on the hypothetical interpretation as well as the facts themselves, and by these it will be amended or rejected. The corpuscular theory represented existing knowledge as regards the propagation of light with sufficient completeness in its day, to be able to indicate the direction of attack for the development of new knowledge and new relations: in so far it had all the utility of a valid scientific hypothesis: but as the science became enlarged the features in which it was unavailing rose into the more prominent place, so much that it became degraded to the position of an analogy reaching only over a portion of the field of phenomena some time before the crucial experimental determinations of the velocity of light in material media decisively robbed it of all higher claim, by proving that in one department of the phenomena its analogy was in error.

It is not superfluous to consider sometimes what there is to prevent many of the scientific hypotheses of physics, chemistry, and other branches of Natural Philosophy, which are at present effective and successful, from being similarly of a merely provisional and analogical character. The uniformities which it is customary to call laws of nature are often just as much laws of mind: they form an expression of the implications between mind and matter, by means of which material phenomena are mentally grasped. The mere effort of the mind after a wider formulation of these implications will not be wholly fortuitous and useless for progress even when it leads temporarily towards error, for that effort is itself an orderly development taking place in the cosmos of interacting mind and matter, of which successive stages must have wider and deeper ramifications than appear on the surface. The formal analogies between the mathematical theories of different branches of physics perhaps originate as much in the nature of the necessary processes of thought as in the nature of the external things: for 'the mind sees in all things that which it brings with it the faculty of seeing.'

43. A glance at the order of historical development of electrical theory will serve for further illustration. Here the point of view under which an exact theory was developed, throughout the greater part of the present century, was that of the various portions of a permanent entity called electricity exerting mutual forces at a distance across empty space, after the analogy of the law of gravitation. This scheme was absolutely complete for all the usual electrostatic applications. In the domain of electrodynamics and magnetism it explained and coordinated, in the hands of Ampère, Neumann, and Weber, a vast range of otherwise extremely complicated phenomena: von Helmholtz and Lord Kelvin showed how it might have anticipated Faraday's cardinal discovery of the electromagnetic induction of electric currents: Kirchhoff found by calculation that according to it waves of very high period would be propagated along a metallic wire with a velocity which according to Weber's fundamental electric determinations comes out to be about the same as the velocity of light. The scheme also, in Weber's hands, gave a definite and rational account of the mechanical attraction between portions of matter carrying electric currents, and between portions of magnetized matter. As elaborated by Weber it was in fact a complete formulation of the whole domain of the experimental electric science of the time: the circumstance that it was insufficient for the case of bodies moving with velocities at all approximating to that of light, or for vibrations with frequency so high as to approach that of light, was unknown because the production of such experimental conditions had not then been attempted, while the continuity between electrodynamic and optical phenomena had only been vaguely guessed at*. So far as existing knowledge went, the only kind of objection to which the Weberian electrodynamics was exposed was a critical attack on its foundations. This was carried out with strong insistence by von Helmholtz: but his arguments perhaps only brought into clear relief the circumstance that when velocities

* Cf. an interesting early appreciation by Maxwell of Weber's theory, in his memoir On Faraday's Lines of Force, *Camb. Phil. Trans.* 1855; *Collected Papers*, I. p. 208.

or vibration-frequencies comparable with those of radiation were contemplated, the Weberian scheme was incomplete, and could not in such extreme cases stand, in the light of general dynamical criticism, without fundamental modification.

There was no experimental knowledge in existence in electrodynamics, previous to Hertz's quite recent classical researches, that could not fairly be collated under the Weberian doctrine: and the preference expressed by Gauss for the notion of an action propagated in time from one moving electric particle to another, instead of a law of instantaneous attraction across space, must be based rather upon his "subjective conviction" as regards the probable nature and fitness of things, and the striving after a view that would lend itself to orderly development into regions beyond the limit of actual experience, than upon any inadequacy of the Weberian type of formula to include and explain all that was then actually known of electrodynamic actions. "In a very interesting letter from Gauss to W. Weber (March 1845) he refers to the electrodynamic speculations with which he had been occupied long before, and which he would have published if he could then have established that which he considered the real keystone of electrodynamics, namely the deduction of the force acting between electric particles in motion from the consideration of an action between them, not instantaneous, but propagated in time, in a similar manner to that of light. He had not succeeded in making this deduction when he gave up his electrodynamic researches, and he had a subjective conviction that it would be necessary in the first place to form a consistent representation of the manner in which the propagation takes place" (Maxwell, 'Treatise,' § 861)*.

* Cf. Appendix D. Two other attempts at theories of propagation, of a different kind, are noticed by Maxwell, 'Treatise,' § 862.

That of Riemann depends on an assumed propagation of an electric potential V according to the formula

$$a^{-2}\, d^2V/dt^2 = \nabla^2 V + 4\pi\rho,$$

ρ being electric density. This is really the equation of propagation of pressure in compressible fluid, in which there is a distribution of sources of strengths amounting to $\rho\,(1 - a^{-2}\, dV/dt)$ per unit volume, or simply of strength ρ when the fluid is nearly incompressible. Though this theory of ideal fluid motion and

44. The consistent representation thus aimed at, of the mode in which electrodynamic action is propagated across free space,—the absence of which formed a barrier to Gauss' progress,—is simply in set terms a dynamical, or, if the term is preferred, an analytical theory of the activity of the luminiferous medium. The solution of that problem was developing along different lines of its own, at the same time as these speculations on laws of electrodynamic action across space were being initiated: a complete analytical scheme of the vibratory activity of the aether, constructed on the basis of a masterly discussion of the optical facts, was actually obtained by MacCullagh in 1839, though he fully admitted that it was not such a solution as had anything in common with analogies of the dynamical propagation of waves across material substances. But the times were not then ripe for a new departure transcending in this way all known material analogy, perhaps owing to the circumstance that the more familiar possibilities could hardly have been considered to be exhausted: and it seems to have been only in the vivid and unconventional intellect of Macquorn Rankine* that the potentialities of MacCullagh's doctrine obtained clear recognition and development. The solution thus given by MacCullagh, of the problem of aethereal constitution, was spelled out through examination of the optical interaction between free aether and aether modified by the presence of matter, isotropic or crystalline: precisely the same solution was independently arrived at by Clerk Maxwell twenty years later through an examination, of quite analogous nature, of the accumulated knowledge of electrical interactions across the aether as modified by the presence of different kinds of matter. Most students would probably be struck by the similarity between the analytical methods and

vibration, with mobile sources and sinks, would lead to interesting hydrodynamic analysis, it cannot afford a sufficiently wide basis on which to construct the much more complex electric theory.

The other attempt, made by Betti (*Nuovo Cimento* 1868), assumes that an electric current is made up of polarized elements like elementary twists in a solid elastic medium. The extent to which such an analogy carries in electro dynamics has been specified, *Phil. Trans.* 1897 A, p. 212.

* *Miscellaneous Scientific Papers*, pp. 63, 160.

mental endowments of these two great analysts: but needless to say Maxwell's speculation is of far wider and more universal scope than MacCullagh's purely optical theory. His efforts to fit the range of physical phenomena into an entirely new analytical frame, involving brilliant strokes of explanation diversified by abrupt transitions and unbridged *lacunae*, his essays at utilizing the analytical machinery of the older method of attractions towards the development of a new descriptive scheme that was to wholly supersede it, impart all the interest of nascent discovery to his 'Treatise.' It would be very wide of the mark to apply to any portion of this great constructive effort his own Miltonic characterization of the thermodynamic investigations in which Rankine "through the palpable obscure finds out his uncouth way"; yet the conception of a struggle with confusion—successful but unfinished—not unfitly expresses the feeling that gains strength on each successive survey of the 'Treatise,' and which mainly arises from the necessary initial imperfections of a reconstruction of ideas in a vast domain in which the natural order and logical precision in exposition have not had time to be elaborated*. In one of Maxwell's appreciations he contrasts the finished electrodynamic exposition of Ampère's 'Essai,' in which all traces of the natural growth of the argument are obliterated in the interests of a severe logical sequence, with the 'Electrical Researches' of his own master Faraday which constitute a continuous history of his mental occupation with the subject, of his governing ideas, his successes and failures as they arose. He was himself to furnish what will perhaps long remain the classical example of the history of an effort of scientific thought of another kind, in which the aim is not so much to enlarge the field of experimental knowledge by the guidance of a few novel master ideas, as to seize upon a domain of knowledge already dimly and obscurely interlaced, and arrange it as parts of a single definite doctrine.

* Cf. for example the difficulties discussed by Hertz, in the introduction to his Collected Electric Papers, as to what Maxwell really intended to be meant by his fundamental conception of electric displacement.

45. *On Vector Terminology.*—The most effective way of avoiding the abuses of merely analogical hypothesis, while at the same time retaining the advantages of vividness of description that belong to it, lies in the development of scientific language. If we can avail ourselves of an adjective that will naturally and precisely connote, and suggest by itself, the similarity of relations that is the subject of an analogy, we shall have effectually annexed its essence while rejecting the superfluous part. The subjects of discussion in mathematical physics may be described in the main as vector quantities in space; and the invention, mainly by Hamilton, by Lord Kelvin and his brother James Thomson, and by Maxwell, of an appropriate language to express the properties of the different classes of vectors, has had much to do with progress in the principles of this science as distinct from computations relating to special problems. It may further conduce to brevity and perspicuity if we keep in view the desirability of rendering the existing terminology still more precise in the department of this field with which we are here concerned.

A vector (u, v, w) will be called a *stream* vector (the *solenoidal* vector of Maxwell) throughout any region in which it satisfies the condition of continuity of flow $du/dx + dv/dy + dw/dz = 0$. There may be singular points in the region at which this condition breaks down through (u, v, w) there becoming infinite: these are the sources and sinks of the flow. If the vector is not a stream, all points in the region will be sources or sinks, so that we can still represent it by a flux of incompressible fluid which soaks out of the region at each point of it, the amount that disappears per unit time in an element of volume $\delta\tau$ being $-(du/dx + dv/dy + dw/dz)\,\delta\tau$; the quantity multiplying $\delta\tau$ is called, after Maxwell, the *concentration* of the vector into the point considered; when the vector is a stream its concentration is null. When (u, v, w) represents flow of heat, its concentration at any point represents the accumulation of heat per unit time per unit volume at the place, and the type of the vector is wholly unrestricted: thus the flow of heat is according to our definition a stream only when there is no accumulation of heat, that is when the temperature is steady. The region of a

stream can be divided up into tubes of flow, each of which has the properties of an independent pipe or channel devoid of leakage. In certain cases—in fact in all the ordinary cases in which the originating disturbance is local and does not involve the creation of new sources—these channels are ringshaped and the flow in them is thus a circulation round the channels: the flow may then be called a *cyclic stream*.

A vector of the type $-(d/dx, d/dy, d/dz)\chi$ it is proposed to call a *gradient* vector (the *simple lamellar* of Maxwell), as it is the gradient or slope of the scalar quantity χ. If this scalar is multiple-valued, its lines of slope will be ring-shaped curves returning into themselves; and the vector may then be called a *cyclic gradient*. The total gradient from one point to another is estimated as a line integral along a path connecting them: its value round a complete circuit back to the point of starting is called, after Lord Kelvin, the *circulation* in that circuit and is null, or else equal to a cyclic constant of the scalar function χ of which the vector is the gradient. A circuit in which there is circulation encloses of necessity, is linked with, a core of some kind around which the circulation is established; and this core must itself be either of infinite length or ringshaped.

The term *circuital*, as introduced by Lord Kelvin, is synonymous with *stream*, thus including cases in which circulation of the stream is not contemplated; it is therefore entirely distinct from *cyclic*.

Aethereal Constitution of Matter

46. The difficulty of imagining a definite uniform limit of divisibility of matter will always be a philosophical obstacle to an atomic theory, so long as atoms are regarded as discrete particles moving in empty space. But as soon as we take the next step in physical development, that of ceasing to regard space as mere empty geometrical continuity, the atomic constitution of matter (each ultimate atom consisting of parts which are incapable of separate existence, as Lucretius held) is raised to a natural and necessary consequence of the new standpoint. We may even reverse the argument, and derive

from the ascertained atomic constitution of matter a philosophical necessity for the assumption of a *plenum*, in which the ultimate atoms exist as the nuclei which determine its strains and motions**.

This idea of a *plenum* with uniform properties throughout all extension, but permeated by intrinsic singular points, each of which determines and, so to speak, locks up permanently a surrounding steady state of strain or other disturbance, forms the ultimate basis of all developments relating to the constitution of aether and matter such as are here attempted.

To make a beginning in the direct or synthetical manner, it is necessary to assign a working scheme of properties to the *plenum*. One way of starting off is to rely on optical theory. The *plenum* must be the medium of transmission of radiation, with its known finite velocity. It must therefore be specified, in dynamical terms, as possessing, when disturbed, energy of strain and energy of inertia; for it is only by the interaction of these that propagation in time can be conceived under a dynamical scheme, which takes account of nothing except substance and motion. The precise formal nature of these endowments of the *plenum* was first unravelled by MacCullagh in his masterly analysis of the optical phenomena of crystals. But he realized very clearly that nothing of the nature of such a type of strain as he was led to postulate, can be thought of as associated with ordinary matter; so he retained his specification of the dynamical constitution of the *plenum* as a purely analytical scheme, that is, as a consistent scheme of properties of this *ultra*-material medium which he could not illustrate from the behaviour of elastic matter. Shortly afterwards Rankine, never timid in his speculations, expounded MacCullagh's analytical scheme soundly and clearly, in full contrast with the elastic properties of matter, as representing a uniform medium or *plenum* endowed with ordinary inertia but with elasticity of purely rotational type. This conception has recently been revived by Lord Kelvin, who illuminated the whole matter by

** It is perhaps not superfluous to point out the argument here involved against any tendency we might have to assign to the aether itself an atomic structure.

showing how by aid of gyrostatic systems the abstract conception of a rotationally elastic medium could be illustrated and closely copied in a material model*.

It is curious that, although the idea of an intimate connexion between the propagation of electric and of optical effects has always been present to speculative physics, yet no attempt was made to ascertain whether MacCullagh's *plenum* could in addition to its vibratory functions take up such a state of permanent strain as would represent the electrostatic actions between charged conductors, or such state of motion as would represent the electrodynamic action between currents. The first hint on this side of the matter was FitzGerald's passing remark in 1880† that MacCullagh's optical equations are identical with those of the electrodynamic theory of optics developed by Maxwell.

47. The basis of the present scientific procedure thus rests on the view, derivable as a consequence of general philosophical ideas, that the master-key to a complete unravelling of the general dynamical and physical relations of matter lies in the fact that it is constituted as a discrete molecular aggregate existing in the aether. At the same time all that is known (or perhaps need be known) of the aether itself may be formulated as a scheme of differential equations defining the properties of a *continuum* in space, which it would be gratuitous to further explain by any complication of structure; though we can with great advantage employ our stock of ordinary dynamical concepts in describing the succession of different states thereby defined.

On account of the very high velocity C of transmission and equilibration of elastic disturbances in the aether, it follows that (on the assumption of a stagnant aether) the motion of material systems across it produces no sensible deviation from the mere succession of equilibrium states of that medium which correspond to the separate configurations of the matter as they arise, so long as the velocity of the matter is not comparable to that of radiation. It is for this reason that simple convection

* Cf. Appendix E.

† 'On the Electromagnetic Theory of Light,' *Phil. Trans.* 1880.

of the electric fields belonging to material bodies furnishes so good a first approximation to the laws of the electrodynamics of bodies in motion: although in some cases, such as unipolar induction due to the spinning of a magnet round an axis of symmetry, care must be taken to realize in forming a physical picture of the phenomenon that the effective moving elements are the separate independent electrons, not material bodies as a whole*.

In the case of a homogeneous body moving with uniform velocity v, there will occur changes in the velocity of radiation across it of the order of the first power of v/C, because this velocity is, like v, a directed phenomenon. But the changes of the scalar properties and dimensions of the body itself are of the order of the square of v/C: for example, it is found as a matter of observation, that the relative free periods corresponding to the spectral lines of gases are not altered to the first order by translatory motion of the vibrating molecules along with the Earth and the Solar System.

Mutual aid of electrical and general molecular theory

48. As thus formulated in terms of the aethereal constitution of the individual atoms, the problem of the aethereal relations of material media is one of molecular dynamics; and it shares in all the difficult and refined considerations of averaging which belong to that branch of physics. But it may be held that its discussion contributes more to the principles of general molecular dynamics than it receives from them. The laws of electrical phenomena have been primarily ascertained in their larger features by a process of mixed induction and deduction, which proceeded, for more than three-quarters of a century, on wholly different lines from those laid down here. These laws, thus independently and in part empirically ascertained, must be derivable from the molecular standpoint; and their demonstration in that manner confirms and vividly illustrates the principles by which a transition is made from the dynamics of systems of discrete molecules to the dynamics of their aggregates treated as continuous matter. Practically the

* Cf. *Phil. Trans.* 1895 A, pp. 727—31.

only field (outside the theory of gases) on which these underlying principles connecting molecular theory with general mechanics have hitherto had scope, is the theory of capillarity; and this constitutes an application that has been held not to be free from difficulty. Now in the problem of a cloud of mutually influencing electrons we are on a clearer basis of physical reality than in a discussion of particles acting on each other with hypothetical forces at a distance obeying undetermined laws. The constitution of an electron is quite definite; and its reaction on its neighbours is quite definite, for the reason that its energy is located in a definite manner in the surrounding aether. Precision reigns everywhere in the *data*; and the transition from the separate electrons to the aggregates forming the material medium, treated as continuous as it is presented to the perceptions of sense, must therefore be a definite logical process capable of explicit and precise formulation.

The theory of the dynamical interaction between the aether and the matter which subsists in it is on a different plane from a mere formal adaptation of the equations which represent the constitution and activity of the free aether to the case where its properties are modified by the presence of matter. Such an empirical adaptation has worked well for the case in which the matter is at rest*: but for the case in which it is moving with velocity yielding appreciable influence on the phenomena, that is with velocity not wholly insensible compared with the speed of radiation, the adaptation in this way has involved the merest guesswork.

The ultimate inadequacy of a method of treating material media, based on merely empirical or speculative additions to the ascertained equations of free aether, had indeed been clearly recognized by von Helmholtz for the last decade of his

* "...And if we attempt to extend our theory [of radiation] to the case of dense media, we become involved not only in all the ordinary difficulties of molecular theories, but in the deeper mystery of the relation of the molecules to the electromagnetic medium. To evade these difficulties we shall assume that in certain media the specific capacity of electrostatic induction is different in different directions,..." Maxwell, 'Treatise,' II § 794.

life. It would appear that one main object of the close scrutiny of the analytical foundations of dynamics, particularly of the single principle of Action which may be made to cover their whole extent, with which he occupied himself during that period, was with a view to arrive at a definite interlaced deduction of the complex of electrodynamic relations from a single analytical function which would express the state of the medium at each instant.

CHAPTER VI

DYNAMICAL THEORY OF ELECTRICAL ACTIONS

Least Action, fundamental in General Dynamics

49. THE idea of deducing all phenomenal changes from a principle of least expenditure of effort or action dates for modern times, as is well known, from the speculations of Maupertuis. The main illustration with which he fortified his view was Fermat's principle of least time for ray propagation in optics. This optical law follows as a direct corollary from Huygens' doctrine that radiation is propagated by wave-motions. In Maupertuis' hands, however, it reverted to the type of a dogma of least action in the dynamical sense as originally enunciated vaguely by Descartes, which Fermat's statement of the principle as one of least time was intended to supersede*; under that aspect it was dynamically the equally immediate corollary of the corpuscular theory of optical rays which was finally adopted by Newton.

The general idea of Maupertuis at once attracted the attention of mathematicians; and the problem of the exact specification of the Action, so as to fulfil the minimum relation, was solved by Euler for the case of orbits of particles. Shortly afterwards the solution was re-stated with greater precision, and generalized to all material systems, by Lagrange (*Mem. Taurin.*, 1760) in one of his earliest and most brilliant memoirs, which constructed the algorithm of the Calculus of Variations, and at the same time also laid the foundation of the fundamental physical science of Analytical Dynamics. The subse-

* Cf. Appendix D.

quent extensions by Hamilton of the Lagrangian analytical procedure involve, so far as interpretation has hitherto been enabled to go, rather fundamental developments in the mathematical methods than new physical ideas,—except in the weighty result that the mere expression of all the quantities of the system as differential coefficients of a single characteristic function establishes relations of complete reciprocity between them, and also between the various stages, however far apart in time, of the system's progress.

It is now a well-tried resource to utilize the principle that every dynamical problem can be enunciated, in a single formula, as a variation problem, in order to help in the reduction to dynamics of physical theories in which the intimate dynamical machinery is more or less hidden from direct inspection. If the laws of any such department of physics can be formulated in a minimum or variational theorem, that subject is thereby virtually reduced to the dynamical type: and there remain only such interpretations, explanations, and developments, as will correlate the integral that is the subject of variation with the corresponding integrals relating to known dynamical systems. These developments will usually take the form of the tracing out of analogies between the physical system under consideration and dynamical systems which can be directly constructed to have Lagrangian functions of the same kind: they do not add anything logically to the completeness and sufficiency of the analytical specification of the system, but by being more intuitively grasped by the mind and of more familiar type, they often lead to further refinements and developments which carry on our theoretical views into still higher and more complete stages.

Derivation of the Equations of the Electric Field from the Principle of Least Action

50. It has been seen (§ 48) that the only effective method of working out the dynamics of molecular systems is to abolish the idea of force between the molecules, about which we can directly know nothing, and to formulate the problem as that of the determination of the natural sequence of changes of

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configuration in the system. If the individual molecules are to be permanent, the system, when treated from the molecular standpoint, must be conservative; so that the Principle of Least Action supplies a foundation certainly wide enough, if only it is not beyond our powers of development.

We require first to construct a dynamical scheme for the free aether when no material molecules are present. It is of course an elastic medium: let us assume that it is practically at rest, and let the vector (ξ, η, ζ) represent the displacement, elastic and other, of its substance at the point (x, y, z) which arises from the strain existing in it. We assume (to be hereafter verified by the results of the analysis) for its kinetic energy T and its potential energy W the expressions

$$T = \tfrac{1}{2}A\int(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)\, d\tau$$

$$W = \tfrac{1}{2}B\int(f^2 + g^2 + h^2)\, d\tau$$

in which $\delta\tau$ denotes an element of volume, A and B are constants, the former a constant of inertia, the latter a modulus of elasticity, and in which (f, g, h) is a vector defined as regards its mode of change** by the relation

$$(\dot{f}, \dot{g}, \dot{h}) = \frac{1}{4\pi}\left(\frac{d\dot{\zeta}}{dy} - \frac{d\dot{\eta}}{dz},\ \frac{d\dot{\xi}}{dz} - \frac{d\dot{\zeta}}{dx},\ \frac{d\dot{\eta}}{dx} - \frac{d\dot{\xi}}{dy}\right), \ldots\text{(I)}$$

where the 4π is inserted in order to conform to the ordinary electrical usage.

This definition makes

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0,$$

so that (f, g, h) is a stream vector.

To obtain the dynamical equations of this medium, we have to develope the variational equation

$$\delta\int(T - W)\, dt = 0,$$

subject to the time of motion being unvaried.

** This allows for the permanent existence, independently of $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$, of the intrinsic aethereal displacement surrounding each electron. Cf. Appendix E.

Now

$$\delta \int T dt = A \int dt \int (\dot\xi \delta\dot\xi + \dot\eta \delta\dot\eta + \dot\zeta \delta\dot\zeta)\, d\tau$$

$$= A \left| \int (\dot\xi \delta\xi + \dot\eta \delta\eta + \dot\zeta \delta\zeta)\, d\tau \right|_{t_1}^{t_2}$$

$$- A \int dt \int (\ddot\xi \delta\xi + \ddot\eta \delta\eta + \ddot\zeta \delta\zeta)\, d\tau.$$

Also

$$\delta W = \frac{B}{4\pi} \int \left\{ f \left(\frac{d\delta\zeta}{dy} - \frac{d\delta\eta}{dz} \right) + g \left(\frac{d\delta\xi}{dz} - \frac{d\delta\zeta}{dx} \right) + h \left(\frac{d\delta\eta}{dx} - \frac{d\delta\xi}{dy} \right) \right\} d\tau$$

$$= \frac{B}{4\pi} \int \{(ng - mh)\, \delta\xi + (lh - nf)\, \delta\eta + (mf - lg)\, \delta\zeta\}\, dS$$

$$+ \frac{B}{4\pi} \int \left\{ \left(\frac{dh}{dy} - \frac{dg}{dz} \right) \delta\xi + \left(\frac{df}{dz} - \frac{dh}{dx} \right) \delta\eta + \left(\frac{dg}{dx} - \frac{df}{dy} \right) \delta\zeta \right\} d\tau.$$

where (l, m, n) is the direction vector of the element of boundary surface δS.

In these reductions by integration by parts the aim has been as usual to express dependent variations such as $\delta\dot\xi$, $d\delta\zeta/dy$, in terms of the independent ones $\delta\xi$, $\delta\eta$, $\delta\zeta$. This requires the introduction of surface integrals: if the region under consideration is infinite space, and the exciting causes of the disturbance are all at finite distance from the origin, these surface integrals over an infinitely remote boundary cannot in the nature of things be of influence on the state of the system at a finite distance, and in fact it may be verified that they give a null result: in other cases they must of course be retained.

On substitution in the equation of Action of these expressions for the variations, the coefficients of $\delta\xi$, $\delta\eta$, $\delta\zeta$ must separately vanish both in the volume integral and in the surface integral, since $\delta\xi$, $\delta\eta$, $\delta\zeta$ are perfectly independent and arbitrary both at each element of volume $\delta\tau$ and at each element of surface δS. This gives, from the volume integral, the equations of vibration or wave-propagation

$$\frac{B}{4\pi} \left(\frac{dh}{dy} - \frac{dg}{dz},\ \frac{df}{dz} - \frac{dh}{dx},\ \frac{dg}{dx} - \frac{df}{dy} \right) = -A\, (\ddot\xi,\ \ddot\eta,\ \ddot\zeta) \text{.... (II)}$$

The systems of equations (I) and (II), thus arrived at, become identical in form with Maxwell's circuital equations which express the electrostatic and electrodynamic working of free aether, if $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ represents the magnetic induction and (f, g, h) the aethereal displacement; the velocity of propagation is $(4\pi)^{-1} (B/A)^{\frac{1}{2}}$, so that $B/A = 16\pi^2 c^2$ where c is the velocity of radiation. They are also identical with MacCullagh's optical equations, the investigation here given being in fact due to him.

51. Now let us extend the problem to aether containing a system of electrons or discrete electric charges. Each of these point-charges determines a field of electric force around it: electric force must involve aether-strain of some kind, as has already been explained: thus an electric point-charge is a nucleus of intrinsic strain in the aether. It is not at present necessary to determine what kind of permanent configuration of strain in the aether this can be, if only we are willing to admit that it can move or slip freely about through that medium much in the way that a knot slips along a rope: we thus in fact treat an electron or point-charge of strength e as a freely mobile singular point in the specification of the aethereal
cf. p. 261 strain (f, g, h), such that very near to it (f, g, h) assumes the form $-\frac{e}{4\pi}\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\frac{1}{r}$. We can avoid the absolutely infinite values, at the origin of the distance r, by treating the nucleus of the permanent strain-form not as a point but as a very minute region*: this analytical artifice will keep all the elements of the integrals of our analysis finite, while it will not affect any physical application which considers the electron simply as a local charge of electricity of definite amount.

Now provided there is nothing involved in the electron except a strain-form, no inertia or energy foreign to the aether residing in its nucleus such as would prevent free unresisted mobility, as it is perhaps difficult to see how there could be, the equations (I) and (II) still determine the state of the field of aether, at any instant, from its state, supposed completely known, at the previous instant: and this determination includes

* This substitution affects only the *intrinsic* molecular energy; cf. *Phil Trans.* 1894 A, pp. 812—3. See Add. & Corr.

a knowledge of the displacement of the nucleus of each strain-form during the intervening element of time. These equations therefore suffice to trace the natural sequence of change in the complex medium thus constituted by the aether and the nuclei pervading it. But if the nuclei had inertia and mutual actions of their own, independent of the aether, there would in addition to the continuous equations of motion of the aether itself be dynamical equations of motion for each strain-form as well, which would interact and so have to be combined into continuity with the aethereal equations, and the problem would assume a much more complex form: in other words, the complete energy function employed in formulating the Principle of Least Action would also involve these other types of physical action, if they existed.

52. But for purposes of the electrodynamic phenomena of material bodies, which we can only test by observation and experiment on matter in bulk, a complete atomic analysis of the kind thus indicated would (even if possible) be useless; for we are unable to take direct cognizance of a single molecule of matter, much less of the separate electrons in the molecule to which this analysis has regard. The development of the theory which is to be in line with experience must instead concern itself with an effective differential element of volume, containing a crowd of molecules numerous enough to be expressible continuously, as regards their average relations, as a volume-density of matter. As regards the actual distribution in the element of volume of the really discrete electrons, all that we can usually take cognizance of is an excess of one kind, positive or negative, which constitutes a volume density of electrification, or else an average polarization in the arrangement of the groups of electrons in the molecules which must be specified as a vector by its intensity per unit volume: while the movements of the electrons, free and paired, in such element of volume must be combined into statistical aggregates of translational fluxes and molecular whirls of electrification. With anything else than mean aggregates of the various types that can be thus separated out, each extended over the effective element of volume,

mechanical science, which has for its object matter in bulk as it presents itself to our observation and experiment, is not directly concerned: there is however another more abstract study, that of molecular dynamics, whose province it is to form and test hypotheses of molecular structure and arrangement, intended to account for the distinctive features of the mechanical phenomena aforesaid.

As the integral $\int(lf + mg + nh)\,dS$, extended over the boundary of any region, no longer vanishes when there are electrons in that region, it follows that the vector (f, g, h) which represents the strain or "electric displacement" of the aether, is no longer circuital when these individual electrons are merged in volume-densities, as they are when we consider a material medium continuously distributed, instead of merely the aether existing between its molecules; thus the definition of the mode of change of aethereal elastic displacement, namely

$$4\pi (\dot{f}, \dot{g}, \dot{h}) = \text{curl } (\dot{\xi}, \dot{\eta}, \dot{\zeta}),$$

which held for free aether, would now be a contradiction in terms. In order to ascertain what is to replace this definition, let us consider the translation of a single electron e from a point P_1 to a neighbouring point P_2. This will cause an addition to the elastic strain (f, g, h) of the aether, represented by a strain-vector distributed with reference to lines which begin at P_1 and end at P_2, the addition being in fact the electric displacement due to the doublet formed by $-e$ at P_1 and $+e$ at P_2. This additional flux of electric displacement from P_2 to P_1 along these lines is not by itself circuital; but the circuits of the flux will be completed if we add to it a linear flux of electricity of the same total amount e, back again from P_1 to P_2 along the line $P_1 P_2$. If we complete in this way the fluxes of *aethereal electric displacement*, due to the changes of position of all the electrons of the system, by the fluxes of these *true electric charges* through the aether, a new vector is obtained which we may call the flux of the *total electric displacement* per unit volume; and this vector forms a fundamentally useful conception from the circumstance that it is everywhere and always a circuital or stream vector.

We may now express this result analytically: to the rate of change of aethereal displacement $(\dot{f}, \dot{g}, \dot{h})\,\delta\tau$ in the element of volume $\delta\tau$ there must be added $\Sigma(e\dot{x}, e\dot{y}, e\dot{z})$, where $(\dot{x}, \dot{y}, \dot{z})$ is the velocity of a contained electron e, in order to get a circuital result: the *current of aethereal electric displacement* by itself is not circuital when averaged with regard to this element of volume, but the so-called *total current*, made up of it and of the *true electric current* formed by the moving electrons, possesses that property.

Thus we have to deal, in the mechanical theory, with a more complex problem: instead of only aethereal displacement we have now two *independent* variables, aethereal displacement, and true electric current or flux of electrons. In the molecular analysis, on the other hand, the minute knowledge of aethereal displacement between and around the electrons of the molecules involved that of the movements of these electrons or singularities themselves, and there was only one independent variable, at any rate when the singularities are purely aethereal. The transition, from the complete knowledge of aether and individual molecules to the averaged and smoothed out specification of the element of volume of the complex medium, requires the presence of two independent variables, one for the aether and one for the matter, instead of a single variable only.

53. We may consider this fundamental explanation from a different aspect. There are present in the medium electrons or electric charges each of amount e, so that for any region Faraday's hypothesis gives

$$\int (lf + mg + nh)\, dS = \Sigma e\,;$$

and therefore, any finite change of state being denoted by Δ, $\Delta\int(lf+mg+nh)\,dS$ is equal to the flux of electrons into the region across the boundary. Thus for example

$$\frac{d}{dt}\int (lf + mg + nh)\, dS = -\int (lu_0 + mv_0 + nw_0)\, dS$$

in which (u_0, v_0, w_0) is the true electric current which is simply this flux of electrons reckoned per unit time: hence

transposing all the terms to the same side, we have for any closed surface

$$\int (lu + mv + nw)\, dS = 0,$$

where $\quad (u, v, w) = (df/dt + u_0,\ dg/dt + v_0,\ dh/dt + w_0).$

This relation expresses that (u, v, w), the total current of Maxwell's theory, is circuital or a stream.

The true current (u_0, v_0, w_0) above defined includes all the possible types of co-ordinated or averaged motions of electrons, namely, currents arising from conduction, from material polarization and its convection, from convection of charged bodies.

54. We have now to fix the meaning to be attached to $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ or (a, b, c) in a mechanical theory which treats only of sensible elements of volume. Obviously it must be the mean value of this vector, as previously employed, for the aether in each element of volume. With this meaning it is now to be shown that the curl of $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ is equal to $4\pi\,(u, v, w)$. We shall in fact see that for any open geometrical surface or sheet S of sensible extent, fixed in space, bounded by a contour s, Sir George Stokes' fundamental analytical theorem of transformation of a surface integral into a line integral round its contour, must under the present circumstances assume the wider form

footnote, 130

$$\frac{1}{4\pi}\Delta\int\left(\xi\frac{dx}{ds} + \eta\frac{dy}{ds} + \zeta\frac{dz}{ds}\right) ds = \Delta\int (l\bar{f} + m\bar{g} + n\bar{h})\, dS + \mathfrak{F} \quad \ldots \text{(i)}$$

where the symbol Δ represents the change in the integral which follows it, produced by the motion of the system in any finite time, and $\mathfrak{F}$ represents the total flux of electrons through the fixed surface S during that time. To this end consider two sheets S and S' both abutting on the same contour s: then as the two together form a closed surface we have *)

$$\int (l'f + m'g + n'h)\, dS' - \int (lf + mg + nh)\, dS = \Sigma e \quad \ldots \text{(ii)}$$

where Σe denotes the sum of the strengths of the electrons included between the sheets: in this formula the direction vectors (l', m', n') and (l, m, n) are both measured towards the

~~*) The [illegible] contains, magnetic terms; see §72.~~

same sides of the surfaces, which for the former S' is the side away from the region enclosed between them. Now if one of these included electrons moves across the surface S' the form of the integral for that surface will be abruptly altered, an element of it becoming infinite at the transition when the electron is on the surface; and this will vitiate the proof of Stokes' theorem considered as applying to the change in the value of that surface integral. But the form of the integral for the other surface, across which the electron has not penetrated, will not pass through any critical stage, and Stokes' theorem will still hold for the change caused in it. That is, for the latter surface the equation (i) will hold good in the ordinary way without any term such as $\mathfrak{F}$; and therefore by (ii), for the former surface, across which electrons are taken to pass, the term $\mathfrak{F}$ as above is involved.

The relation of Sir George Stokes, thus generalized, in which $\mathfrak{F}$ represents the total flux of electrons across the surface S, leads directly to the equation

$$\operatorname{curl}(\dot\xi, \dot\eta, \dot\zeta) = 4\pi(\dot f + u_0, \dot g + v_0, \dot h + w_0).$$

where the vectors *now* represent mean values throughout the element of volume.

This relation holds, whether the system of molecules contained in the medium is *magnetically* polarized or not, for the transference of magnetic polarity across the sheet S cannot add anything to the electric flux through it: it appears therefore that in a case involving magnetic polarization $(\dot\xi, \dot\eta, \dot\zeta)$ represents what is called the magnetic induction and not the magnetic force, which is also in keeping with the stream character of the former vector.*) On the other hand the change in the *electric* polarization (f', g', h') of the molecules constitutes an addition $\Delta(f', g', h')$ of finite amount per unit area to the flux through the sheet, so that $d/dt\,(f', g', h')$ constitutes a part of the true electric current (u_0, v_0, w_0).

55. It has been seen that the specification of sensible electric motions in a material body involves both the flux of the electrons and the averaged disturbance of the aether as independent variables. In ordinary electrodynamic phenomena

*) The current then contains magnetic terms; see §73.

relating to currents of conduction it is the former that we are by far the more directly concerned with. We have therefore for purposes of ordinary electrodynamics to transform the kinetic energy

$$T = \tfrac{1}{2}A\int(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)\, d\tau, \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ represents magnetic induction as above, into a form which expresses it as the effect of motion of the electrons.

This can be done most easily by introducing, after Maxwell's manner, a subsidiary vector (F, G, H) such that

$$\left(\frac{dH}{dy} - \frac{dG}{dz},\ \frac{dF}{dz} - \frac{dH}{dx},\ \frac{dG}{dx} - \frac{dF}{dy}\right) = (\dot{\xi}, \dot{\eta}, \dot{\zeta}), \quad\ldots\ldots(2)$$

which is permissible on account of the circuital character of $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ or (a, b, c). Then we have

$$T = \tfrac{1}{2}A\int\left\{\dot{\xi}\left(\frac{dH}{dy} - \frac{dG}{dz}\right) + \dot{\eta}\left(\frac{dF}{dz} - \frac{dH}{dx}\right) + \dot{\zeta}\left(\frac{dG}{dx} - \frac{dF}{dy}\right)\right\} d\tau$$

$$= \tfrac{1}{2}A\int\{(n\dot{\eta} - m\dot{\zeta})F + (l\dot{\zeta} - n\dot{\xi})G + (m\dot{\xi} - l\dot{\eta})H\}\, dS$$

$$+ 2\pi A\int(Fu + Gv + Hw)\, d\tau, \quad\ldots\ldots\ldots\ldots\ldots(3)$$

since by the above

$$\left(\frac{d\dot{\zeta}}{dy} - \frac{d\dot{\eta}}{dz},\ \frac{d\dot{\xi}}{dz} - \frac{d\dot{\zeta}}{dx},\ \frac{d\dot{\eta}}{dx} - \frac{d\dot{\xi}}{dy}\right) = 4\pi\,(u, v, w) \quad\ldots\ldots(4)$$

where (u, v, w) is the total current $(\dot{f} + u_0,\ \dot{g} + v_0,\ \dot{h} + w_0)$.

Combining (2) and (4) we have

$$\nabla^2 F - \frac{d}{dx}\left(\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz}\right) = -4\pi u$$

with two similar equations: these are solved by the relations

$$(F, G, H) = \left(\int\frac{u}{r}\, d\tau + F_0,\ \int\frac{v}{r}\, d\tau + G_0,\ \int\frac{w}{r}\, d\tau + H_0\right),$$

wherein (F_0, G_0, H_0) is determined so as to satisfy the system of equations

$$\nabla^2(F_0, G_0, H_0) = \left(\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz}\right)\left(\frac{dF_0}{dx} + \frac{dG_0}{dy} + \frac{dH_0}{dz}\right),$$

of which the most general solution is

$$(F_0, G_0, H_0) = \left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\Phi$$

where Φ is an arbitrary function.

This part of the auxiliary potential (F, G, H), depending on Φ, adds nothing to the variable (ξ, η, ζ) which represents the actual phenomenon, and therefore may be omitted. If it were retained it would mean that something hitherto unspecified, besides the motion of the aether, was fundamentally in operation. Substituting then our result

$$(F, G, H) = \int (u, v, w)\, r^{-1} d\tau,$$

in which any magnetism that may be present is implicitly included as molecular current-whirls*, we have

$$T = 2\pi A \iint (u_1u_2 + v_1v_2 + w_1w_2)\, r_{12}^{-1}\, d\tau_1\, d\tau_2$$

$$= 4\pi A \Sigma\Sigma\, (u_1u_2 + v_1v_2 + w_1w_2)\, r_{12}^{-1}\, \delta\tau_1\, \delta\tau_2,$$

the summation taking each pair of elements $\delta\tau_1$, $\delta\tau_2$ only once, the double integral taking them twice. Here the total current (u, v, w) is made up of the drift of the electrons and the time-rate of change in the electric displacement (f, g, h) of the aether: thus we have expressed the kinetic energy in terms of these quantities. The potential energy W is already expressed in terms of the same variables by the formula

$$W = \tfrac{1}{2} B \int (f^2 + g^2 + h^2)\, d\tau.$$

The auxiliary quantity (F, G, H), which proved to be the potential of the circuital current-vector (u, v, w), can now if it is thought fit be dispensed with. Its use was to facilitate an integration by parts, which collected together those elements of kinetic energy from all over the field that are, in the equation of Action, associated with the electric flux in the element of volume $\delta\tau$.

* The surface-integral terms in (3) corresponding to any interface now vanish on account of the continuity of (F, G, H). For the completion of this analysis for the case of *magnetic* material media, see the end of Appendix A.

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56. The dynamical equation expressing the sequence of events in the system is $\delta\int(T-W)\,dt=0$, with the time not subject to variation: we are now prepared to develope the variation with respect to the system of independent variables composed of the flux of the electrons and (f, g, h): for this purpose we must revert to the complete expression for T.

The part of $T/4\pi A$ which involves the single electron e moving with velocity $(\dot{x}, \dot{y}, \dot{z})$ is, by (3),

$$\tfrac{1}{2}Le^2(\dot{x}^2+\dot{y}^2+\dot{z}^2)+e\dot{x}F+e\dot{y}G+e\dot{z}H,$$

where L is a quantity which our present analysis does not determine*, depending as it does on the size and constitution of the nucleus of the electron.

We might now insert a sign to represent summation over all the electrons and conduct the variation, were it not for the circumstance that our variables are not wholly independent; the variation of (f, g, h) is in fact restricted by the condition

$$\int(lf+mg+nh)\,dS=\Sigma e$$

or

$$\int\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right)d\tau-\Sigma e=0.$$

Hence we must introduce into the variational equation a Lagrangian undetermined function of position Ψ, so that it is the variation of

$$\frac{1}{4\pi A}\int(T-W)\,dt+\int dt\left\{\int\Psi\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right)d\tau-\Sigma e\Psi\right\}$$

that is to be made null; afterwards determining the form of Ψ to satisfy the restriction which necessitated its introduction.

57. Now as regards an electron e, $(4\pi A)^{-1}\,\delta\int T\,dt$ gives the terms

$$\int dt\,Le^2(\dot{x}\delta\dot{x}+\dot{y}\delta\dot{y}+\dot{z}\delta\dot{z})$$
$$+\int dt\left\{e\dot{x}\left(\frac{dF}{dx}\delta x+\frac{dF}{dy}\delta y+\frac{dF}{dz}\delta z\right)+\ldots+\ldots\right\}$$
$$+\int dt\,(eF\delta\dot{x}+eG\delta\dot{y}+eH\delta\dot{z}),$$

* Cf. *Phil. Trans.* 1894 A, pp. 812—3. See Add. & Corr.

which are equal to terms at the time limits together with

$$\int dt\, \delta x \left\{- Le^2\ddot{x} - e\left(\frac{\delta F}{dt} - \dot{x}\frac{dF}{dx} - \dot{y}\frac{dG}{dx} - \dot{z}\frac{dH}{dx}\right)\right\}$$

$$+ \int dt\, \delta y \{\ldots\} + \int dt\, \delta z \{\ldots\},$$

where $\frac{\delta}{dt}$ denotes $\frac{d}{dt} + \dot{x}\frac{d}{dx} + \dot{y}\frac{d}{dy} + \dot{z}\frac{d}{dz}$, because the time-integral refers to a travelling electron; thus finally giving terms at the time limits together with

$$\int dt\, \delta x \left[-Le^2\ddot{x} + e\left\{\dot{y}\left(\frac{dG}{dx} - \frac{dF}{dy}\right) - \dot{z}\left(\frac{dF}{dz} - \frac{dH}{dx}\right) - \frac{dF}{dt}\right\}\right]$$

$$+ \int dt\, \delta y\, [\ldots] + \int dt\, \delta z\, [\ldots].$$

As regards the variation of the state of the free aether, represented by (f_2, g_2, h_2) in an element of volume $\delta\tau_2$ containing no electrons, we have in $(4\pi A)^{-1}\,\delta\int T dt$ the terms

$$\delta \int dt\, \Sigma\Sigma\, \{(u_0 + \dot{f})\dot{f}_2 + (v_0 + \dot{g})\dot{g}_2 + (w_0 + \dot{h})\dot{h}_2\}\, r^{-1}\, \delta\tau\, \delta\tau_2.$$

Of this the part involving the variation of the aethereal electric displacement in the single element of volume $d\tau$ (written in place of $\delta\tau_2$ in order to avoid subscripts) gives

$$d\tau\,.\,\delta \int dt\, (F\delta\dot{f} + G\delta\dot{g} + H\delta\dot{h})$$

which is equal to terms at the time limits together with

$$-\, d\tau\,.\int dt \left(\frac{dF}{dt}\,\delta f + \frac{dG}{dt}\,\delta g + \frac{dH}{dt}\,\delta h\right).$$

Here two points are to be noticed. First, it would not have been correct simply to vary $(4\pi A)^{-1}\int T dt$, involving

$$\tfrac{1}{2}\int dt \int (F\dot{f} + G\dot{g} + H\dot{h})\, d\tau,$$

unless we bore in mind that (F, G, H) itself involves implicitly the independent variable (f, g, h) to be varied. Secondly, in

writing here $d/dt\,(F, G, H)$ it is implied that the translation of the aether itself is negligibly small compared with that of the electrons of matter that are moving through it. Strictly, if (p', q', r') were the velocity of the aether itself we should have $\delta'/dt\,(F, G, H)$ instead of $d/dt\,(F, G, H)$, where δ'/dt would represent $d/dt + p'd/dx + q'd/dy + r'd/dz$: this would introduce enormous complication into the electrodynamic equations, destroying their linearity in a way such as occurs for instance in hydrodynamics. We shall find that our present course fits in with all the electrodynamic phenomena; it is also in keeping with the fact that the most refined optical experiments have been unable to detect translatory movement in the aether.

The reduction of the remaining terms of the variational equation is given by the formulae

$$\frac{1}{4\pi A}\,\delta\int W dt = \frac{B}{4\pi A}\int dt\int (f\delta f + g\delta g + h\delta h)\,d\tau;$$

$$\delta\int dt\int \Psi\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right)d\tau$$

$$= \int dt\int \Psi\,(l\delta f + m\delta g + n\delta h)\,dS$$

$$- \int dt\int\left(\frac{d\Psi}{dx}\,\delta f + \frac{d\Psi}{dy}\,\delta g + \frac{d\Psi}{dz}\,\delta h\right)d\tau;$$

$$\delta\int dt\,\Sigma e\Psi = \int dt\,\Sigma e\left(\frac{d\Psi}{dx}\,\delta x + \frac{d\Psi}{dy}\,\delta y + \frac{d\Psi}{dz}\,\delta z\right).$$

58. The variations δx, δy, δz which give the virtual displacement of an electron e, and the variations δf, δg, δh which specify the electric displacement of a point in the free aether, can now be considered as all independent and perfectly arbitrary: hence the coefficient of each must vanish separately in the dynamical variational equation. Thus we obtain two sets of equations, of types

$$-\frac{B}{4\pi A}f - \frac{dF}{dt} - \frac{d\Psi}{dx} = 0$$

$$-Le^2\ddot{x} + e\left(\dot{y}\zeta - \dot{z}\eta - \frac{dF}{dt} - \frac{d\Psi}{dx}\right) = 0.$$

If as before we write $(4\pi C)^2$ for B/A, these equations become

$$4\pi C^2 f = -\frac{dF}{dt} - \frac{d\Psi}{dx}$$

$$Le^2 \ddot{x} = e\left\{\dot{y}\zeta - \dot{z}\eta - \frac{dF}{dt} - \frac{d\Psi}{dx}\right\};$$

they are the differential equations which determine the sequence of events in the system. Expressed in the ordinary language of electrodynamics, which avails itself of the conception of force, they show that

$$-dF/dt - d\Psi/dx$$

is the x component of the *aethereal force* which strains the free aether; and that

$$\dot{y}\zeta - \dot{z}\eta - dF/dt - d\Psi/dx$$

is the x component of the *electric* force which tends to accelerate the motion of an electron e. Each electron has an effective mass Le^2, of aethereal origin, which forms part and may be the whole of the mass of the matter to which it is attached. As previously explained*, the real advantage of thus introducing the conception of forces is that we can develope methods of reasoning about actual systems by attaching to each sensibly permanent portion of the material system the forces that are invariably associated with it, thereby promoting in the domain of dynamics the comparisons of relations on which all logical processes are based.

59. We have just spoken, in order to avoid complex reservations, of the electric force acting on the single electron e: in strictness this is of course more than our analysis gives us. The equation which is thus interpreted should strictly have a Σ, a sign of summation, in front of it, to show that it is an aggregate equation for all the electrons in the element of volume with which we are in reality dealing. It will appear that in the subsequent sorting out of the different kinds of electric motion that occur in the element of volume, no harm will ensue from the present mode of expression, if we now attend to one point.

In certain cases a sensible part of the electric force acting on the single electron arises from the other electrons in the

* Cf. also Appendix B.

same element of volume: in such cases, however, when we express what we are really concerned with, namely the summation throughout the element of volume, this action may be cancelled by a complementary reaction, so that in the aggregate such terms will not remain. Now in the expression above given for the electric force acting on an electron this case arises when the medium is magnetized. It has been shown that the vector $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ that occurs in the expression for this force is the magnetic induction (a, b, c): and in the theory of magnetism it is shown* that of this induction a part (α, β, γ) called the magnetic force arises from the system in general, and the remainder $4\pi(A, B, C)$ is the expression of local influence arising in the element of volume itself. The question arises then whether the latter part is to be rejected in effecting the summation over the element of volume, as compensated by reaction exerted by the electron under consideration on the magnetism existing in the same element of volume. If it is a question of finding the mechanical force acting on the complete element of volume this compensation will subsist: the action of the magnetism in the element on the electron will just cancel the action of the electron on the magnetism. But if it is a question of finding the electric force which produces a current by separating positive and negative electrons in the element, no such compensation will occur, because the reaction of the electron on the magnetism has no connexion with such electric separation.

Thus we have, for our mechanical theory which considers only elements of volume, the expressions for the aethereal force (P', Q', R') and electric force (P, Q, R) as follows:

$$P = \dot{y}c - \dot{z}b - \frac{dF}{dt} - \frac{d\Psi}{dx}$$

$$P' = -\frac{dF}{dt} - \frac{d\Psi}{dx}$$

$$= P - \dot{y}c + \dot{z}b,$$

where $(F, G, H) = \int (u, v, w)\, r^{-1} d\tau$ in which expression the magnetism is to be included as molecular current-whirls,

* Cf. Appendix A: also p. 108, footnote.

leading to the expressions of § 66 ††, while Ψ is a function of position which has to be determined so as to ensure that the total current is circuital. We may say that the reaction arising from the constraint against non-circuital flow, which is involved in the constitution of the aether, is represented by a term in the aethereal force and also by a term in the electric force, both derived from the same potential Ψ. And it is to be remembered that in computing the mechanical force per unit volume, (a, b, c) must be replaced in these equations by (α, β, γ)**.

It now remains to complete our analysis of the sequence of events in the medium in bulk, by classifying the various kinds of motions of electrons which are connected with this electric force, each according to its own law.

Specification and Relations of a Current of Conduction

60. First let us take the case of a current of conduction flowing in a linear circuit. It must be made up of a drift of electrons, or ions, positive ones travelling in one direction, negative ones in the opposite direction, under the influence of the electric force. There must be as many negative as positive in the element of volume: if not, the element would be electrified, and there could not be a steady electric state until the excess which constitutes the free charge is driven to the surface of the conductor.

The aggregate of all these ions is acted on by the electric force (P, Q, R); as there are as many negative as positive the last two terms in the expression above for the electric force give a null aggregate on summation for all of them. The first two terms however give a force acting on the element of volume, equal to $(\gamma\Sigma e\dot{y} - \beta\Sigma e\dot{z},\ \alpha\Sigma e\dot{z} - \gamma\Sigma e\dot{x},\ \beta\Sigma e\dot{x} - \alpha\Sigma e\dot{y})$

†† For the reduction of these terms, see Appendix A, or *Phil. Trans.* 1895 A, p. 816.

** The expressions for T and W, which form the basis of this analysis, involve only electric terms: if it should prove necessary to include other terms as well, there would of course be other forces and actions arising from them, in addition to the electric ones here obtained.

By definition $(\Sigma e\dot{x}, \Sigma e\dot{y}, \Sigma e\dot{z}) = (u_0, v_0, w_0)\,\delta\tau$, where (u_0, v_0, w_0) is the *true current* or electric flow per unit volume including convection as well as conduction: hence this force is $(v_0\gamma - w_0\beta, w_0\alpha - u_0\gamma, u_0\beta - v_0\alpha)$ per unit volume of the material medium: it is the mechanical force of electrodynamic origin (§ 65) acting on a conductor or other material body carrying a current.

61. We have now to examine the character of the relation between the current of conduction (u', v', w') and the electric force that drives it by urging the positive electrons or ions one way and the negative ones the opposite way. The velocity of drift of each of these kinds of ions among the molecules of the metallic conductor is in the steady state proportional to the electric force: but there is as yet nothing to show whether these velocities are equal and opposite. In any case however the aggregate drift, forming the current of conduction, must be proportional to the electric force; say

$$(u', v', w') = \sigma | (P, Q, R),$$

where the $|\sigma|$ indicates that, if the medium is not isotropic, σ will be a vector coefficient.

But we have a source of information as to whether the velocities of the positive and negative electrons are equal and opposite. We shall first consider ions, as in electrolysis, that is sub-atoms which each contain one or more uncompensated electrons, so that positive and negative ions have usually different masses: we know that in electrolysis they have also different facilities of migration through the solvent fluid, as we should expect. But the law of Faraday asserts that notwithstanding this tendency for the one to travel faster than the other the amounts of current delivered by the two are the same, as shown by the fact that the amounts of the ions that are liberated at the two electrodes are electrochemical equivalents. Now this restriction to equality can only arise from some constraint imposed on the electrolyte from outside, in this case therefore from the metallic part of the circuit. The fact that all electrolytes are fluid makes it reasonable to assume that the ions in a solid are not freely and independently mobile; so that conduction in a solid would rather take place

by something of the nature of the passing on of the same ion or electron successively through different molecules which do not themselves migrate. If we assume that conduction in a metal is something akin to this, the number of ions liberated per unit time at the positive end must of necessity be always the same as that liberated at the negative end; and must be the same as the number that cross any intermediate section of the conductor. Thus we have grounds for the conclusion that the current in a metal is carried, in the Grotthus fashion without diffusion of ions, half by positive and half by negative ions: and therefore perforce this division also holds good in steady flow in any circuit of which part is metallic. But in a circuit wholly electrolytic the different mobilities of the various ions are not thus under control, especially if the solutions are dilute; and this ratio of equality need not be preserved*.

Indeed even the original Williamson-Clausius hypothesis of transient occasional dissociation involves that such dissociation shall become complete and permanent, when the molecules are very sparsely scattered through a foreign medium so that there is not much chance of immediate recombination: while when the molecules are very densely distributed, a very slight amount of very transient dissociation is all that need occur, especially if the ions are very mobile as in metals, in fact is all that chemical knowledge as to molecular permanence permits us to assume.

Specification of a Polarization Current

62. When a molecule is electrically polarized to moment M, a displacement of positive electrons has occurred towards one end of it and of negative towards the other end such that Σed, the sum formed by adding the product of each electron and its displacement in the direction of the resultant moment, is equal to M. Thus dM/dt is equal to $\Sigma e\,dd/dt$ for each molecule. This, being a vector statement, is true for each component of

* Cf. Appendix C.

the polarization separately: and summing for all the molecules per unit volume, we have thus

$$\frac{d}{dt}(f',\ g',\ h') = (\Sigma e\dot{x},\ \Sigma e\dot{y},\ \Sigma e\dot{z}),$$

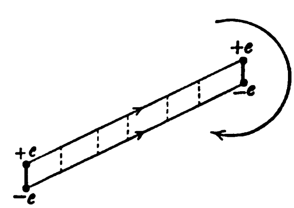
where $(\dot{x}, \dot{y}, \dot{z})$ is the velocity of the electron e. That is, there is a true electric flux per unit volume arising from change of polarization of the material, which is specified by

$$\frac{d}{dt}(f',\ g',\ h').$$

Specifications of the Currents of Electric Convection arising from the motion of charged and of polarized Material Media

63. A material medium moving with velocity equal at the point (x, y, z) to (p, q, r), and having in the neighbourhood of that point a charge of electrons amounting to ρ per unit volume, clearly contributes a convection current $(p, q, r)\rho$.

The convection of a material medium merely polarized to intensity (f', g', h') also supplies a part to the volume distribution of electric current: but its determination requires more refined analysis. Consider in the first place the convection of a simple type of polar molecule involving a single electron $+e$ for one pole and another $-e$ for the other pole. The transfer of these two electrons in company, as in the diagram, is equivalent to the transfer of a positive electron round the long narrow circuit in the direction of the curved arrow: and this circuit can be divided up into sub-circuits of ordinary form in the Amperean manner by partitions represented by the dotted lines. The distance between the two poles of the molecule is absolutely negligible compared with the distance that the molecule is carried by the convection, in a time which is effectively infinitesimal for the analytical theory of continuous currents even in its optical applications: so that the circumstance that convection round the ends of the elongated circuit is not really effected is immaterial. It follows



that the convection with velocity (p, q, r) of a medium, containing such molecules polarized or orientated to intensity (f', g', h'), gives rise to an Amperean system of currents round minute circuits, which forms effectively a magnetic polarization, or *quasi*-magnetism of the material, of intensity $(rg'-qh', ph'-rf', qf'-pg')$ per unit volume. This result will clearly not be disturbed when the distribution of polarity in the molecule is more complicated than that here assumed for the purpose of explanation.

64. It will be convenient in most cases to retain this mode of specification by means of a distribution of magnetization, as it will enable us to take direct advantage of the known principles governing such distributions. But we can restore it to the form of a distribution of currents. Consider in fact any bodily magnetization (A, B, C); and taking the component A separately, let us represent it by Amperean current-whirls with their planes at right angles to the axis of x. Expand the areas of these whirls, and diminish their intensities in the same proportion, until their contours come into contact, thus forming a network in the plane. On summing up the currents in an element of volume we now readily obtain, combining the opposite flows along each branch of the network, dA/dz parallel to y and $-dA/dy$ parallel to z, per unit volume, together with an uncompensated flow round the external contour of the network. Thus on superposition for all three components, we find that the magnetization (A, B, C) is equivalent to a bodily distribution of current $\left(\frac{dC}{dy}-\frac{dB}{dz}, \frac{dA}{dz}-\frac{dC}{dx}, \frac{dB}{dx}-\frac{dA}{dy}\right)$ together with a current sheet on the bounding surface. The precise range of properties for which this equivalence holds good will be presently investigated: for some analytical purposes it is very convenient to convert in this manner all the magnetism and *quasi*-magnetism associated with the medium into a volume-distribution of electric currents. We may call the aggregate electric flux (u_1, v_1, w_1) obtained by including all this the total effective current: thus, when magnetism and *quasi*-magnetism, of aggregate intensity (A_1, B_1, C_1), is present, and when the surface of the magnet is replaced by

a gradual transition so that the current sheet on it is absorbed in the bodily current, we have

$$u_1 = u' + \frac{df''}{dt} + \frac{dC_1}{dy} - \frac{dB_1}{dz} + p\rho,$$

$$F = \int \frac{u_1}{r}\, d\tau,$$

where $f'' = f + f'$, and $A_1 = A + rg' - qh'$.

The complete Mechanical Force acting on the Material Medium

65. The force of electrodynamic origin acting on the matter in bulk is the aggregate of the forces acting on its electrons, according to the formula of § 59 which gives a force on an electron e whose x component is

$$e\left(\dot{y}\gamma - \dot{z}\beta - \frac{dF}{dt} - \frac{d\Psi}{dx}\right), \quad \text{or} \quad e\,(P' + \dot{y}\gamma - \dot{z}\beta):$$

this makes up in all a static part $\Sigma eP'$

and a kinetic part $\Sigma e\dot{y}\gamma - \Sigma e\dot{z}\beta$.

The static part is equal to

$$\rho P' + f'\frac{dP'}{dx} + g'\frac{dP'}{dy} + h'\frac{dP'}{dz},$$

of which the first term arises from the aggregate of uncompensated free ions and the second from the aggregate of polarized molecules. The kinetic part contains terms arising from the drift of free ions constituting the current of conduction, and the differential drift of combined electrons constituting the polarization current, and the convection of free electric charges, making up in all

$$\gamma(v - \dot{g}) - \beta(w - \dot{h}),$$
$$q(w - \dot{h}) - r(v - \dot{g}),$$

where (p, q, r) is the velocity of the element of volume of the matter and $(u - \dot{f}, v - \dot{g}, w - \dot{h})$ is the total current of Maxwell less the purely aethereal part which is not of the nature of electric flux: this part of the force contains also a portion arising from the orbital motions of electrons in the molecules,

which constitute when polarized the magnetism of the medium, with which can be conveniently included the *quasi*-magnetism arising from the convection of electrically polarized molecules, thus giving in all

$$A_1 \frac{da}{dx} + B_1 \frac{da}{dy} + C_1 \frac{da}{dz},$$

where the magnetic force (α, β, γ) is *defined* to be $(a - 4\pi A_1, b - 4\pi B_1, c - 4\pi C_1)$, in which as above $A_1 = A + rg' - qh'$.

Thus we have in all for the mechanical bodily force (X, Y, Z) an expression of type

$$X = \left(v - \frac{dg}{dt}\right)\gamma - \left(w - \frac{dh}{dt}\right)\beta + A_1 \frac{da}{dx} + B_1 \frac{da}{dy} + C_1 \frac{da}{dz}$$
$$+ f' \frac{dP'}{dx} + g' \frac{dP'}{dy} + h' \frac{dP'}{dz} + \rho P'.$$

This expression is in agreement with Ampère's results for the simple case of an ordinary current of conduction, giving a mechanical force at right angles to each current element. It has been shown* that a formula for the electrodynamic energy which involves the aggregate current per unit volume only, and not the individual electrons, cannot lead to correct results in this respect: its basis is too narrow for the facts.

The part of the component X that depends on the motion of the matter is

$$\rho\,(q\gamma - r\beta) + (rg' - qh')\frac{da}{dx} + (ph' - rf')\frac{da}{dy} + (qf' - pg')\frac{da}{dz}.$$

For example when a transparent isotropic body conveying plane-polarized electric waves along the z axis with their magnetic vector along the x axis and their electric vector along the y axis, is moving with velocity (p, q, r), its elements sustain alternating mechanical force arising from its motion, in the direction of the magnetic vector and equal to $-pg'da/dz$, or $-p(\mu^2 - 1)dE'/dt$, where E' is the statical part $KQ^2/8\pi C^2$ of the radiant energy per unit volume and μ^2 is equal to the dielectric constant K, while p is the component of the velocity of the medium in the direction of this force.

* *Phil. Trans.*, 1895 A, pp. 698—701.

On the specification of a Magnetic Distribution in terms of a continuous distribution of Electric Currents

66. It appears from the molecular analysis that the electric vector potential (F, G, H) arising from a magnetic distribution (A, B, C) is given at points outside the magnetism by equations of the type

$$F = \int \left(B \frac{d}{dz} - C \frac{d}{dy} \right) \frac{1}{r} \, d\tau ;$$

but that at places inside the magnetism

$$F = \left| \int (nB - mC) \frac{1}{r} \, dS \right|_1^2 + \int \left(\frac{dC}{dy} - \frac{dB}{dz} \right) \frac{1}{r} \, d\tau ,$$

the previous formula being then plainly inapplicable because it integrates to a quantity whose differential coefficients are indefinite when r can vanish*. Analogously, it may be recalled that, in the ordinary statical theory of magnetism, the magnetic force is derived from the potential of the actual magnetic polarity only at places outside the magnet, but at places in its interior is derived from the potential of the Poisson volume and surface distributions of an ideal continuous magnetic substance. At a point in the interior of the magnetism the magnetic force should be in fact *defined* as the part of the force, acting on a unit pole there situated, that is independent of the local polarity at the spot, it being then shown how the definite value of this part can be determined.

In all such cases the definition of the quantity concerned, thus amended so as to give a definite finite value at points in the interior of the magnetic system, extends of course to the exterior as well. But at exterior points the whole of the polarity is efficient, there being no local effect to be omitted: thus this transformation from a polarity to a volume and surface distribution would as regards outside points serve

* Cf. Appendix A. It may however be immediately verified that the latter is the correct form by its satisfying the relation curl $(F, G, H) = (a, b, c)$ which formed the definition of the vector potential. [The above amounts to saying that the actual discrete distribution of magnetism must be replaced by an averaged distribution which is continuous not only as regards itself but also as regards its gradient.]

no useful purpose, and only obscure the real nature of the formulae.

67. It is of importance to define the scope of the equivalence thus indicated by the formula for the vector potential between (i) a magnetic distribution (A, B, C), and (ii) an electric current distribution equal to

$$\left(\frac{dC}{dy}-\frac{dB}{dz},\ \frac{dA}{dz}-\frac{dC}{dx},\ \frac{dB}{dx}-\frac{dA}{dy}\right)$$

throughout the volume together with sheets of current given by

$$\Big|\,(nB-mC,\quad lC-nA,\quad mA-lB)\,\Big|_1^2,$$

flowing along the interfaces.

We notice that this bodily distribution of current satisfies the equation of continuity of flow, and is therefore everywhere a *stream*. If we represent each interface between different media as a gradual but very rapid transition, throughout which our volume integration has play, there will be no surface sheets to be attended to; and this will often be a great simplification, because the surface current-sheet is not usually a stream. Consider in fact any flat volume element $\delta S\,\delta n$, of thickness δn, of the layer of transition: the current which flows out along the layer is fed by the flow into it through its opposite faces δS, hence the current in the layer is a stream only when there is no resultant flow into the layer across its faces, that is, only when the flux denoted by curl (A, B, C) is continuous on the two sides of it, which will not usually be the case. This current-sheet is therefore an example of a *flux* which is not a *stream*.

The equivalence between these two systems, one a magnetic and the other a current system, includes *ex hypothesi* that of the vector potential and therefore of its curl, that is of the quantity $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ which occurs in the dynamical analysis as the aethereal disturbance and is always a stream vector. What is this quantity, in terms of the usual magnetic conceptions? We have

$$\frac{d\dot{\zeta}}{dy}-\frac{d\dot{\eta}}{dz}=-\nabla^2F+\frac{dJ}{dx}$$

where J represents $dF/dx + dG/dy + dH/dz$ and is always null by the formula for (F, G, H). But $\nabla^2 F = -4\pi\left(\frac{dC}{dy} - \frac{dB}{dz}\right)$. Hence curl $(\dot{\xi}, \dot{\eta}, \dot{\zeta}) = 4\pi$ curl (A, B, C); so that $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ is made up of the gradient of some continuous potential together with $4\pi(A, B, C)$. Outside the magnetism the potential gradient thus introduced can be none other than the magnetic force (α, β, γ) of the magnetic system, as ordinarily defined: and a representation may be introduced which will extend its definition to the interior, in the usual manner, by formation of an ideal cavity. Thus we have

$$(\dot{\xi}, \dot{\eta}, \dot{\zeta}) = (\alpha, \beta, \gamma) + 4\pi(A, B, C);$$

which shows that $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ represents the magnetic induction of the magnetized system.

It thus appears that these electric and magnetic systems are equivalent as regards magnetic induction. They are not however equivalent as regards magnetic force; for in the one case the curl of the magnetic force is 4π times the current, in the other it is null. In treating of a current system devoid of magnetism, the only quantity that occurs is the magnetic induction due to the currents: the portion of the expression for this induction which forms the contribution of the part of the current arising from contiguous molecules or elements of volume being *always* negligible compared with the induction as a whole. The magnetic force is thus not one of the primary quantities of electrodynamic theory as here developed on the single basis of moving electrons**: it is a concept introduced by the transition from molecular dynamics to mechanical theory, being the mean aethereal disturbance $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ diminished by the part arising from purely local causes.

** It may be well to recall here that the magnetism is actually constituted of permanent currents of electric convection, of molecular dimensions, made up of the orbital motions of electrons that are involved in the constitution of the molecule.

CHAPTER VII

REVIEW OF THE ELECTRODYNAMIC EQUATIONS OF A MATERIAL MEDIUM

Exact Dynamical Relations

68. In each case only one relation typical of the set of three equations will be set down. The equations marked by Roman numerals are of the nature of definitions of the new quantities that occur in them: the others are dynamical relations. We have

$$4\pi c^2 f = P' = -\frac{dF}{dt} - \frac{d\Psi}{dx} \qquad (1)$$ p. 97, 98

$$P = P' + qc - rb \qquad (2)$$ p. 98, 102

where
$$a = \frac{dH}{dy} - \frac{dG}{dz}; \qquad (i)$$ p. 92

also we have the total effective current (u_1, v_1, w_1) given by

$$u_1 = u' + \frac{df''}{dt} + \frac{dC_1}{dy} - \frac{dB_1}{dz} + p\rho \qquad (3)$$ p. 104

where
$$f'' = f + f' \qquad (ii)$$ 〃

$$A_1 = A + rg' - qh'; \qquad (iii)$$ 〃

also
$$F = \int \frac{u_1}{r} d\tau \qquad (4)$$ 〃

$$X = \left(v - \frac{dg}{dt}\right)\gamma - \left(w - \frac{dh}{dt}\right)\beta + A_1\frac{da}{dx} + B_1\frac{da}{dy} + C_1\frac{da}{dz}$$
$$+ f'\frac{dP'}{dx} + g'\frac{dP'}{dy} + h'\frac{dP'}{dz} + \rho P' \qquad (5)$$ p. 105

where $$u - \frac{df}{dt} = u_0 = u' + \frac{df'}{dt} + p\rho \ldots\ldots\ldots\ldots\ldots \text{(iv)}$$ p.104

$$\alpha = a - 4\pi A_1. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}$$ p.105

Exact Relations inherent in the Constitution of the Medium

69. We have $$\rho = \frac{df''}{dx} + \frac{dg''}{dy} + \frac{dh''}{dz} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$-\frac{\delta'\rho}{dt} = \frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Here ρ represents the density of the true electrification,—that is of the unpaired electrons distributed throughout the medium which must have come there by conduction or convection from without: also

$$\frac{\delta'\rho}{dt} \text{ denotes } \frac{d\rho}{dt} + \frac{dp\rho}{dx} + \frac{dq\rho}{dy} + \frac{dr\rho}{dz},$$

being the rate at which the electric charge in a given *material element* changes with the time, account being of course taken of change of form and position of the element. The second of these relations, namely equation (7), expresses the fact that such change can take place only by conduction of electrons through the material medium from element to element: convection can merely transfer the volume electrification *along with* the material element in which it occurs.

The question arises whether the relation of the conduction current to the electric force is altered by motion of the material medium which is the seat of the conduction. As the current is made up half of the positive electrons urged one way by the electric force and half of the complementary negative ones urged the opposite way, it follows that any influence of the motion of the medium on the one half is neutralized by its influence on the other half, unless it be an influence involving the *square*, or higher even powers, of the velocity of the medium. Hence the coefficients of conductivity of a medium are altered, by motion of the conductor, at most only to the order of the square of the ratio of its velocity to the velocity of radiation.

Further exact Relations deduced from the above

70. It follows from (3) combined with (6) and (7) that

$$\frac{du_1}{dx}+\frac{dv_1}{dy}+\frac{dw_1}{dz}=0. \quad\text{.....................(8)}$$

Thus the total effective current, as well as the Maxwellian total current, is always a stream vector, like the flow of an incompressible fluid.

This relation combined with (4) gives

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0 \quad\text{.....................(9)}$$

Thus the vector potential of the electrodynamic system is also always a stream vector.

On substituting from (1) and (i) in (6) we obtain, by aid of (9),

$$-\frac{1}{4\pi C^2}\nabla^2\Psi=\rho-\left(\frac{df'}{dx}+\frac{dg'}{dy}+\frac{dh'}{dz}\right) \quad\text{..........(10)}$$

As the right-hand side is equal to $\rho+\rho'$, where ρ' is the Poisson density of ideal electrification which is the equivalent (for certain purposes, *cf.* Appendix A) of the electric polarization, it follows that Ψ, originally introduced into the analysis as an undetermined multiplier, is always the static electric potential of a distribution of density $\rho+\rho'$, or say ρ'', where ρ'' is what Maxwell called the density of the 'apparent electrification' of the medium. This result, that a term explicitly occurring in the formula for the electric force is in fact the static force due to all the electrification and polarity in the field, in their actual situations at the moment, is in one respect remarkable. It looks at first glance as if this static potential Ψ were propagated instantaneously from the distant parts of the field: but there is really nothing in the analysis to support such a view, any more than there is to suggest the view that the vector potential of the current system given by (4) is instantaneously propagated from all parts of the field; we might also just as well say the same of the Action belonging to an element of volume, which includes contributions from all parts of the

system. In deducing the dynamical relations from the principle of Action, which involves the interactions of the whole system, it has merely been found desirable, for analytical simplification, to introduce these auxiliary functions Ψ, F, G, H which do not directly represent physical quantities. But it will appear that by the suitable analytical procedure, namely by the operation of integrating round linear circuits, we shall be able to eliminate these potentials, and all the necessary relations will be expressible solely in terms of the physical quantities that are propagated, without the aid of auxiliary mathematical conceptions. *)

Relations which express with more or less approximation, as the result of observation and experiment, the physical properties of the Material Medium

71. In any given material medium, devoid of hysteretic quality, the intensity of electric polarization (f', g', h') must be a mathematical function of the electric force (P, Q, R) which excites it: it is the electric force (P, Q, R) and not the aethereal force (P', Q', R') that is thus operative, because it is the former that acts on the electrons of the matter, the latter being on the other hand the forcive*) of type requisite to call out the complementary aethereal elastic displacement (f, g, h). In ordinary cases, certainly in all cases in which the exciting force is small, the relation between (f', g', h') and (P, Q, R) is a linear one: thus in the general problem of an aeolotropic medium there will be nine static dielectric coefficients. The principle of negation of perpetual motions requires this linear relation to be self-conjugate, and so reduces these nine coefficients to six; we can arrive at a knowledge of their values only by experimental determination for the case of each substance. In the special case of isotropy there is only one coefficient, and the relation may be expressed in the usual form

a Kelvin, .. Phil. .. III p. 457

$$(f', g', h') = \frac{K-1}{4\pi C^2}(P, Q, R),$$

where K is the single dielectric constant of the medium.

*) The potentials, combined in a different manner, may indeed be expressed as directly propagated from the sources of the aether [redacted]tance, which are the stationary and moving electrons: see p. 227, added infra.

In problems relating to moving material media, the question arises at the threshold whether the value of K for the medium is sensibly affected by its movement through the aether. When it is considered that each molecule that is polarized by the electric force has effectively two precisely complementary poles, positive and negative, it becomes clear that a reversal of the motion of the material medium cannot alter the polarity induced: hence the influence of the motion on K can depend only on the square and higher even powers of the velocity. Thus the effect of motion of a material medium on its dielectric constant, or on other coefficients of electric polarization, is of the order of the square of the ratio of the velocity of the medium to the velocity of radiation.

In cases in which the magnetization induced in the medium is of sufficient magnitude to be taken into account, similar statements will apply to it. In the general crystalline medium there are six independent coefficients of magnetization: these reduce for an isotropic medium to a single coefficient specified by κ, the magnetic susceptibility, or by μ, the magnetic permeability, as defined by the equations

$$(A, B, C) = \kappa(\alpha, \beta, \gamma) = \frac{\kappa}{\mu}(a, b, c),$$

$$\mu = 1 + 4\pi\kappa.$$

And, as before, motion of the material medium does not alter κ or μ except to the second order of the ratio of the velocity of the medium to the velocity of radiation.

A simple equation of this kind, representing linear and reversible magnetization, applies to substances such as iron only when the field is of small intensity. As to the constancy of the dielectric susceptibility in strong fields exact data are wanting, but it seems likely that there also effects of hysteresis will be developed. In such cases, any empirical law of polarity that is found to suit the case sufficiently, including usually a constant term representing permanent polarization, can take the place of these linear relations.

72. Finally, the relation between the current of conduction (u', v', w') and the electric force may be taken as a linear one

involving in the general case nine independent coefficients of conductivity: in the case of isotropy these reduce to a single coefficient. In the general case this relation, of type

$$u' = \sigma_1 P + \sigma_{12} Q + \sigma_{13} R$$
$$v' = \sigma_{21} P + \sigma_2 Q + \sigma_{23} R$$
$$w' = \sigma_{31} P + \sigma_{32} Q + \sigma_3 R,$$

may be written, after Sir George Stokes, in the form

$$u' = \frac{dD}{dP} + \tfrac{1}{2}(\sigma_{12} - \sigma_{21})\, Q - \tfrac{1}{2}(\sigma_{31} - \sigma_{13})\, R$$
$$v' = \frac{dD}{dQ} + \tfrac{1}{2}(\sigma_{23} - \sigma_{32})\, \overset{R}{\cancel{Q}} - \tfrac{1}{2}(\sigma_{12} - \sigma_{21})\, \overset{P}{\cancel{R}}$$
$$w' = \frac{dD}{dR} + \tfrac{1}{2}(\sigma_{31} - \sigma_{13})\, \overset{P}{\cancel{Q}} - \tfrac{1}{2}(\sigma_{32} - \sigma_{23})\, \overset{Q}{\cancel{R}},$$

where $D = \tfrac{1}{2}\sigma_1 P^2 + \tfrac{1}{2}\sigma_2 Q^2 + \tfrac{1}{2}\sigma_3 R^2$

$$+ \tfrac{1}{2}(\sigma_{23} + \sigma_{32})\, QR + \tfrac{1}{2}(\sigma_{31} + \sigma_{13})\, RP + \tfrac{1}{2}(\sigma_{12} + \sigma_{21})\, PQ.$$

The part of the current depending on the function D is related in a scalar manner to three principal axes of conduction in the crystalline medium, those namely for which the product terms do not occur in D. But the remaining part of the current is at each point at right angles to the electric force, and to an axis fixed in the material medium with its direction vector proportional to $(\sigma_{23} - \sigma_{32},\ \sigma_{31} - \sigma_{13},\ \sigma_{12} - \sigma_{21})$; and it is in magnitude proportional to the component of the electric force perpendicular to this axis, the direction of the flow being determined by a screw rule with reference to a definite assigned direction along the axis taken as the standard one. Thus the existence of terms of this latter type in the equations of conduction implies the presence of a directed quality of some kind, related to this axis, which affects the conduction: this may for example be of the nature of crystalline hemihedry, or it may arise from the influence of an imposed extraneous magnetic field (Hall effect), or conceivably (though not according to the present theory) from a translatory motion of the material medium through the aether. But in all ordinary crystalline and other media, in which the constitution of the

molecule is simply dipolar, so that directed quality whether intrinsic or imposed from without is absent, the nine coefficients of conduction will, as in the previous cases, reduce to six.

It has been found by experiment that coefficients of electric conduction, unlike the other coefficients above considered, remain constant for all intensities of the current up to very high limits, so long as the temperature and physical condition of the conducting substance are not altered. This is what was perhaps to be anticipated from the circumstance that conduction arises from the filtering of the simple non-polar electrons or ions through the conducting medium under the directing action of the electric force, not from orientation of polar complex molecules which may originate hysteretic changes in their cohesive grouping in the substance.

Elimination of Mathematical Potentials: scheme expressed in terms of the Circuital Relations

73. It follows from the formula for (P, Q, R) that

$$\frac{dR}{dy}-\frac{dQ}{dz}=-\frac{\delta a}{dt}+\left(a\frac{d}{dx}+b\frac{d}{dy}+c\frac{d}{dz}\right)p-a\left(\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}\right).$$

This is the analytical expression of Faraday's circuital relation that the line-integral of *electric force* round any circuit which is *carried along with the matter* is equal to the time-rate of diminution of the magnetic induction through it**. When the velocity (p, q, r) of the material medium is uniform in direction and magnitude, it becomes simply

$$\frac{dR}{dy}-\frac{dQ}{dz}=-\frac{\delta a}{dt}. \quad \text{.....................(I)}$$

Again, since (F, G, H) is a stream vector,

$$\frac{dc}{dy}-\frac{db}{dz}=-\nabla^2 F$$

$$=4\pi\left(u+\frac{dC_1}{dy}-\frac{dB_1}{dz}\right).$$

** This relation is universally valid, the amount of tubes of induction cut across by the motion of an element of the circuit being represented by the element of the line-integral of the first terms in the electric force.

Hence, introducing the *definition* expressed by the equations

$$(\alpha, \beta, \gamma) = (a - 4\pi A_1,\ b - 4\pi B_1,\ c - 4\pi C_1) \text{ ...... (i)}$$

we have
$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi u; \text{ .... (II)}$$

this is the expression of Ampère's circuital relation that the line-integral of magnetic force round any circuit, *fixed or moving*, is at each instant equal to the flow of the Maxwellian total current through it multiplied by 4π. It is important to notice that as (α, β, γ) is here introduced into the theory it is a subsidiary quantity defined in terms of (a, b, c) and the magnetization. In this magnetization moreover,—not now in the current for what was called the total effective current (u_1, v_1, w_1) does not appear in this mode of formulation—is included the electrodynamic equivalent of the convection of electric polarization: thus

$$A_1 = A + rg' - qh', \text{ ....(ii)}$$

where, assuming the possibility of permanent magnetization (A_0, B_0, C_0) in addition to magnetization induced according to a linear vector coefficient $|\kappa|$, we have

$$(A, B, C) = (A_0, B_0, C_0) + |\kappa|(\alpha, \beta, \gamma). \text{ ....(1)}$$

To these circuital relations (I) and (II), in which all potential functions have disappeared because from their nature such functions give null results on integration round linear circuits, we have only to add the specification of the electric current, namely

$$u = u' + \frac{df''}{dt} + p\rho, \text{ ....(iii)}$$

where
$$(u', v', w') = |\sigma|(P, Q, R) \text{ .... (2)}$$

$$f'' = f + f' \text{ ....(iv)}$$

$$f = \frac{1}{4\pi C^2}(P - qc + rb) \text{ .... (III)}$$

$$(f', g', h') = (f_0', g_0', h_0') + \left|\frac{K-1}{4\pi C^2}\right|(P, Q, R), \text{ ....(3)}$$

the term (f_0', g_0', h_0') representing any permanent electric polarization that may exist, as for example, in pyro-electric

crystals. A complete scheme of electromotive equations, involving only the physical changes that are propagated, is thus obtained. When the motion of the medium is uniform, the final equations of propagation, however heterogeneous the medium may be, are expressible in terms of either the electric force (P, Q, R) or the magnetic induction (a, b, c). When the motion is not uniform, so that the circuital relation (I)*) requires modification, the latter will be the more convenient set of independent variables.

As the equations of this complete scheme have been here numbered, a capital Roman numeral represents an exact relation of pure dynamics, a small Roman numeral indicates a relation of mere definition, and an Arabic numeral a relation, more or less exact, depending in a manner more or less empirical on the constitution of the material medium.

74. When the material medium, however heterogeneous, is at rest in the aether, these electromotive equations reduce precisely to Maxwell's scheme, of type

$$\frac{dR}{dy} - \frac{dQ}{dz} = -\frac{da}{dt},$$

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi u,$$

$$u = u' + \frac{df''}{dt},$$

$$(u', v', w') = |\sigma| (P, Q, R),$$

$$(f'', g'', h'') = (f_0', g_0', h_0') + (4\pi C^2)^{-1} |K| (P, Q, R),$$

$$a = \alpha + 4\pi A,$$

$$(A, B, C) = (A_0, B_0, C_0) + |\kappa| (\alpha, \beta, \gamma),$$

in which the coefficients $|\kappa|$, $|K|$ may vary from point to point of the medium, and in which permanent magnetic and electric polarizations (A_0, B_0, C_0) and (f_0', g_0', h_0') have for the sake of completeness been retained.

When the material medium is in motion these equations are modified in the following respects: (α) relation (I) is further altered unless the motion is one of uniform translation: (β) there

*) The true current is under all circumstances defined by the flux of electrons across fixed interfaces. At the same time it constitutes a volume distribution $(u_1, v_1, w_1)\,\delta\tau$, as supra p. 102.

is a new term, arising from convection of electric polarization, added to the magnetism, which changes A to A_1 as in (ii): (γ) there is the current arising from convection of electric charge which supplies the term $p\rho$ in (iii), a term which Maxwell in some connexions temporarily overlooked, but which has already been fully restored by FitzGerald and others: (δ) there is the modification in the expression for the total electric displacement that is involved in the terms containing the velocity which occur in (III). Of these changes (β) and (δ) are jointly required, if we are to obtain the correct value for the influence of material convection on the velocity of radiation,—they are thus experimentally verified in so far as one verification can include two relations: (γ) is partially verified by Rowland's experiments on the electrodynamic effect of electric convection, while the influence of (β) has also been to some extent directly detected in a similar manner by Röntgen. When the material system moves without change of form, the influence of (α), which is involved in Maxwell's equations of electric force, extends only to a very slight redistribution of electric charges.

Maxwell's original Electromotive scheme determinate for the case of Media at rest

75. In Maxwell's development of electromotive equations (the equations of mechanical force being excluded) for media maintained at rest, the indeterminateness which arose from the deficiencies of experimental knowledge, was represented, when the effective current was by his main hypothesis of circuitality confined to flow in complete circuits, by the presence of the unknown potential Ψ, contributing to the electric force a part which integrates to nothing round each complete circuit. When this potential is got rid of by elimination, the scheme assumes the form of circuital relations. It follows directly from the form of these relations that, starting from any given initial condition of the system, they suffice to determine uniquely its subsequent course: thus, given the initial distribution of the vectors (P, Q, R) and (α, β, γ), the circuital relations express directly the initial distribution of their time-gradients $d/dt\,(P, Q, R)$

and $d/dt\,(\alpha, \beta, \gamma)$, and therefore also the values of the vectors themselves at the succeeding instant of time; and so on, step by step, for all successive instants of time. This proves that, when the material media are at rest, the potential Ψ thus introduced is not an independent variable, and is in fact not necessary for the electrodynamic problem at all; that its value, if for any purpose it is desired, is implicitly involved in the distribution of electric and magnetic forces alone,—the distribution of electric charge and polarity, of which Ψ is then the potential, being determinable directly from the concentration of the electric force.

76. *Electric Theory of Double Refraction: comparison with von Helmholtz's Generalization.*—To illustrate the bearing of this result, we shall consider Maxwell's investigation of optical propagation in a crystalline medium at rest, referred to its principal dielectric axes. The equations are

$$(P,\ Q,\ R) = -\left(\frac{dF}{dt} + \frac{d\Psi}{dx},\ \frac{dG}{dt} + \frac{d\Psi}{dy},\ \frac{dH}{dt} + \frac{d\Psi}{dz}\right),$$

where Ψ is an undetermined function which might originally, as Maxwell constructed the theory, have been supposed to involve possible phenomena hitherto unelucidated:

$$(u,\ v,\ w) = (4\pi C^2)^{-1}\frac{d}{dt}(K_1P,\ K_2Q,\ K_3R),$$

where K_1, K_2, K_3 are the inductive capacities in the directions of the principal electric axes, which are chosen as axes of reference: while

$$\operatorname{curl}(F,\ G,\ H) = (a,\ b,\ c), \quad \operatorname{curl}(\alpha,\ \beta,\ \gamma) = 4\pi\,(u,\ v,\ w):$$

and we take the medium to be non-magnetic so that

$$(a,\ b,\ c) = (\alpha,\ \beta,\ \gamma).$$

The latter equations lead to

$$\begin{aligned}\nabla^2 F - \frac{dJ}{dx} &= -4\pi u \\ &= K_1 C^{-2}\left(\frac{d^2F}{dt^2} + \frac{d^2\Psi}{dx\,dt}\right),\end{aligned}$$

where J represents $dF/dx + dG/dy + dH/dz$. In Maxwell's indirect way of introducing (F, G, H) this vector is made up of a definite part due to the magnetism, a definite part due to the current, and an undetermined part such as must not affect the integrated electric force round a complete circuit: we can arrange (and therefore ought to so arrange in order to avoid the danger arising from too many variables apparently but not really independent) that the latter part is absorbed in Ψ so that (F, G, H) becomes definite, this definite expression making J null identically since the current is circuital. Now in order to develope from these equations the circumstances of electric disturbances travelling in plane sheets, let F, G, H and Ψ all be functions of $lx + my + nz - Vt$, or say of ω, so that V is the velocity of propagation of a disturbance travelling in the direction (l, m, n): on substitution in the electrodynamic equations, writing (a^2, b^2, c^2)** for $C^2(K_1^{-1}, K_2^{-1}, K_3^{-1})$, we obtain

$$\frac{d^2F}{d\omega^2} = \frac{V^2}{a^2}\frac{d^2F}{d\omega^2} - \frac{lV}{a^2}\frac{d^2\Psi}{d\omega^2},$$

that is

$$\frac{d^2F}{d\omega^2} = \frac{lV}{V^2 - a^2}\frac{d^2\Psi}{d\omega^2},$$

with similar expressions for $d^2G/d\omega^2$ and $d^2H/d\omega^2$. The relation J null or

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0$$

then gives for the velocity of propagation Fresnel's equation

$$\frac{l^2}{V^2 - a^2} + \frac{m^2}{V^2 - b^2} + \frac{n^2}{V^2 - c^2} = 0,$$

the alternatives V null, or Ψ* independent of ω, being irrelevant as referring to a steady state.

** This notation will not here be confused with that for magnetic induction.

* The investigation as given by Maxwell in 'Treatise,' §§ 794—7, is vitiated by making Ψ vanish, apparently by an oversight: that quantity is the static potential of the polarization of the material medium and so travels along with the waves. It does however vanish when the medium is isotropic, in so far as it is not a stationary potential due to permanent electric charge: cf. § 36.

In the electric propagation the current (u, v, w) and the magnetic induction (a, b, c) are both, quite irrespective of the character of the medium, in the plane of the wave-front, simply because they are both circuital: and they are moreover at right angles to each other on account of the curl relation. This may be at once verified by referring the disturbance for a moment to coordinate axes one of which is at right angles to the wave-front.

To determine the direction (λ, μ, ν) of the current vector in the wave-front, we have

$$-K_{,}C^{-2}\frac{dP}{dt} = \nabla^2 F = \frac{d^2F}{d\omega^2},$$

so that

$$-4\pi u = \frac{lV}{V^2 - a^2}\frac{d^2\Psi}{d\omega^2};$$

hence

$$\lambda : \mu : \nu = \frac{l}{V^2 - a^2} : \frac{m}{V^2 - b^2} : \frac{n}{V^2 - c^2}.$$

Thus the electric vector, and therefore also the magnetic vector, is parallel to one of the principal axes of the section of Fresnel's ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ by the plane of the wave-front.

77. It may be remarked that Maxwell's own investigation (*loc. cit.*) is apparently more general than the above in that it shows that the same law of velocity of propagation would hold good if in the final equations of propagation

$$dF/dx + dG/dy + dH/dz$$

did not vanish identically, but were denoted by a new variable J, the reason in fact being that Ψ, on which J depends, travels along with (F, G, H). Substituting F, G, H, Ψ as functions of ω in the equations of propagation, then of type

$$\nabla^2 F - \frac{dJ}{dx} = K_1 C^{-2}\left(\frac{d^2F}{dt^2} + \frac{d^2\Psi}{dx\,dt}\right),$$

we would then have

$$\left(1 - \frac{V^2}{a^2}\right)\frac{d^2F}{d\omega^2} = l\frac{dJ}{d\omega} - \frac{Vl}{a^2}\frac{d^2\Psi}{d\omega^2},$$

with similar equations for G, H: so that

$$\frac{dJ}{d\omega} = \left(\frac{a^2l^2}{a^2 - V^2} + \frac{b^2m^2}{b^2 - V^2} + \frac{c^2n^2}{c^2 - V^2}\right)\frac{dJ}{d\omega}$$

$$- \left(\frac{l^2}{a^2 - V^2} + \frac{m^2}{b^2 - V^2} + \frac{n^2}{c^2 - V^2}\right) V \frac{d^2\Psi}{d\omega^2},$$

or
$$\left(\frac{dJ}{d\omega} - \frac{1}{V}\frac{d^2\Psi}{d\omega^2}\right)\left(\frac{l^2}{a^2 - V^2} + \frac{m^2}{b^2 - V^2} + \frac{n^2}{c^2 - V^2}\right) = 0.$$

Thus the alternatives would be that the waves are propagated according to Fresnel's law of velocity, or that the state of disturbance is one in which there is no electric force.

The situation is revealed at a glance on transforming the equations of propagation from the potential (F, G, H) to the electric force (P, Q, R) as independent variable: they then become of type

$$\nabla^2 P - \frac{d}{dx}\left(\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz}\right) = \frac{1}{a^2}\frac{d^2P}{dt^2},$$

in which J and Ψ disappear simultaneously: thus waves of electric force are propagated according to the same laws whether J is taken to vanish or not. In either case the electric force (P, Q, R) is not itself in the plane of the wave-front, but the vector $(a^{-2}P, b^{-2}Q, c^{-2}R)$ is so and corresponds to Fresnel's radiation-vector.

In his discussion of the Maxwellian dielectric scheme from the point of view of action at a distance, von Helmholtz started from the assumption of a definite generalization of the Neumann electrodynamic energy formula, which led him to a scheme in which the current was not circuital, and thence to electric condensational waves propagated with a definite finite velocity, while there were still waves of exactly transverse type propagated with the velocity given by Maxwell's law. It has been noticed that this result is much wider than von Helmholtz's special hypothesis*. The restriction to circuital currents however at once excludes any such condensational waves.

* *Roy. Soc. Proc.* XLIX., 1891, p. 532.

In Maxwell's original analysis of double refraction (*Phil. Trans.*, 1866), the possibility of magnetic aeolotropy was included, with the same principal axes however as the electric aeolotropy: the wave-surface then appeared in the form of Fresnel's surface as modified by homogeneous shear made up of uniform shrinkages on the directions of its principal axes. The interesting remark has been made by Heaviside, and also by A. McAulay, that the wave-surface would still be Fresnel's surface subjected to a homogeneous strain if the magnetic and electric axes of the crystal were different. In fact refer the equations of the circuital system to the magnetic axes: by altering (x, y, z) in certain ratios, and also the components of the various vectors, the equations readily assume with these new variables the form suitable to magnetic isotropy, with its corresponding Fresnel wave-surface: hence transforming back again, the statement is proved. This simple correspondence does not however extend to the dynamical laws of phenomena such as reflexion.

On the Transition from Molecular to Molar or Mechanical Theory

78. A definite and consistent scheme of electrodynamic equations has thus been obtained by regarding the material system as made up of discrete molecules, involving in their constitutions orbital systems of electrons, and moving through the practically stagnant aether. It is not necessary, for the development of the equations, to form any notion of the constitution of the electron, or of how its translation through the aether can be intelligibly conceived. But, inasmuch as the absence of disturbance of the aether by its motion, or by the motion of a system of such electrons, when viewed in the light of the disturbance of a material medium produced by motion of material bodies through it, has often led to an attitude of entire agnosticism with reference to aethereal constitution, it seems desirable that a kinematic scheme* explaining or illustrating the phenomena, such as may be based on the conception of a rotationally elastic aether, should have

* Cf. Appendix E.

a place in the foundations of aether-theory. Any hesitation, resting on *à priori* scruples, in accepting as a working basis such a rotational scheme, seems to be no more warranted than would be a diffidence in assuming the atmosphere to be a continuous elastic medium in treating of the theory of sound. It is known that the origin of the elasticity of the atmosphere is something wholly different from the primitive notion of statical spring, being in fact the abrupt encounters of molecules: in the same way the rotational elastic quality of the incompressible aether, which forms a sufficient picture of its effective constitution, may possibly have its origin in something more fundamental that has not yet even been conceived. But in both cases what is important for immediate practical applications is a condensed and definite basis from which to develope the interlacing ramifications of a physical scheme: and both in the theory of sound and the theory of radiation this is obtained by the use of a representation of the action of the medium which a deeper knowledge may afterwards expand, transform, and even modify in detail. Although however it is possible that we may thus be able ultimately to probe deeper into the problem of aethereal constitution, just as the kinetic theory has done in the case of atmospheric constitution, yet there does not seem to be at present any indication whatever of any faculty which can bring that medium so near to us in detail as our senses bring the phenomena of matter: so that from this standpoint there is much to be said in favour of definitely regarding the scheme of a continuous rotationally elastic aether as an ultimate mode of physical representation.

79. A formal scheme of the dynamical relations of free aether being thus postulated after the manner of Maxwell and MacCullagh, and a notion as clear as possible having been obtained of the aethereal constitution of a molecule and its associated revolving electrons, by aid of the kinematic rotational hypothesis, it remained to effect with complete generality the transition between a molecular theory of the aethereal or electric field which considers the molecules separately, and a continuous theory expressed by differential equations which

take cognizance only of the properties of the element of volume, the latter alone being the proper domain of mechanical as distinct from molecular science. This transformation has been accomplished by replacing summations spread over the distribution of molecules by continuous integrations spread over the space occupied by them. In cases where the integrals concerned all remain finite and definite, when the origin to which they refer is inside the matter so that the lower limit of the radius vector is null, there is no difficulty in the transition: this is for example the case in the domain of the ordinary theory of gravitational forces. But in important branches of the electric theory of polarized media some of the integral expressions became infinite under these circumstances; which was an indication that it is not legitimate to replace the effect of the part of the discrete distribution of molecules which is adjacent to the point considered by that of a continuous material distribution*. The result of the integration still, however, gave us a valid estimate of the effect of the material system *as a whole,* when we bore in mind that the infinite or rather undetermined term entering at the inner limit really represents the part of the result which depends *solely* on the local molecular configuration, a part whose actual magnitude could be determined only when that configuration is exactly assigned or known. The consideration of this indeterminate part is altogether evaded by means of the general mechanical principle of mutual compensation of molecular forcives. This asserts that in such cases, when a sensible portion of the effect per molecule arises from the action of the neighbouring molecules, that part must be omitted from the account in estimating the *mechanical* effect on an element of volume of the medium; indeed otherwise mechanical theory would be impossible††. The mutual, statically equilibrating, actions of

* Cf. Appendix A.

†† In the language of pure mathematics the integrals or rather summations expressing the forcives are divergent at the lower limits; which does not matter for purposes of mechanical theory because it is only their principal values in Cauchy's sense that are there involved.

In case the mechanical and molecular terms in the complete energy function of the material medium were not independent, a mechanical disturbance would

adjacent molecules are effective towards determining the structure of the material medium, and any change therein involves change in its *local* physical constants and properties, which may or may not be important according to circumstances: but such local action contributes nothing towards polarizing or straining the element of mass whose structure is thus constituted, and therefore nothing to mechanical excitation, unless at a place where there is abrupt change of density.

affect its molecular structure and so lead to change of its constitution: such are cases in which pressure alters, gradually or suddenly, the state of dissociation, or of aggregation, of a gas. All such cases are beyond the limits of rational mechanics. The independence between mechanical and molecular theory is therefore not a principle that can be demonstrated theoretically by any process of averaging over the molecules, because it is not always true: it is an inference from the molecular stability and permanence of each special system to which it applies.

CHAPTER VIII

OPTICAL AND OTHER DEVELOPMENTS RELATING TO ENERGY AND STRESS

Mechanical Forcive expressed in terms of Stress

80. IT follows from the analysis of Maxwell, 'Treatise,' ii § 640, that in cases in which the polarization current is insignificant so that $(u - \dot{f},\ v - \dot{g},\ w - \dot{h})$ is practically the same as $(u,\ v,\ w)$, the mechanical electrokinetic forces acting on any portion of a material system *at rest* would be statically equivalent to a traction over its boundary specified as follows, $\mathfrak{H}$ denoting magnetic force and $\mathfrak{B}$ magnetic induction: (i) a hydrostatic pressure $\mathfrak{H}^2/8\pi$, (ii) a tension along the bisector of the angle ϵ between $\mathfrak{H}$ and $\mathfrak{B}$, of intensity $\mathfrak{H}\mathfrak{B}\cos^2\epsilon/4\pi$, and (iii) a pressure along the bisector of the supplementary angle of intensity $\mathfrak{H}\mathfrak{B}\sin^2\epsilon/4\pi$—were it not for an outstanding bodily torque of intensity $\mathfrak{H}\mathfrak{B}\sin 2\epsilon/4\pi$ tending to rotate from $\mathfrak{B}$ towards $\mathfrak{H}$. When $\mathfrak{B}$ and $\mathfrak{H}$ are in the same direction, as they are in isotropic media not permanently magnetized, the torque vanishes, and this representation of the mechanical electrokinetic forces in the form of a stress is perfect: the traction on any surface that arises from the stress is then a hydrostatic pressure $\mathfrak{H}^2/8\pi$ together with a tension $\mu\mathfrak{H}^2/4\pi$ along the lines of magnetic force.

In a similar manner we can derive from the theory of a polarized dielectric the result that the mechanical electrostatic forces, for a system involving only isotropic dielectrics, are derivable from a stress consisting of a hydrostatic pressure $\mathfrak{E}^2/8\pi C^2$ together with a tension $K\mathfrak{E}^2/4\pi C^2$ along the lines of

the electric force $\mathfrak{E}$, to which is however to be added a pressure $(K-1)\mathfrak{E}^2/8\pi C^2$ acting over the surface of each conducting region. These mechanical forces differ from the ones derivable from the Faraday-Maxwell type of electrostatic stress, according to which the function of a *uniform* dielectric is merely to transmit the forces without adding anything to them: since we here regard the material dielectric as polarizable analogously to a magnet, it will be more than a mere medium of transmission as regards the mechanical force.

Repulsion of conducting masses by Magnetic Alternators

81. An elegant application of these results is to the case of oscillatory or alternating electric currents in conductors, of wave-length long compared with the linear dimensions of the material system so that the influence of the polarization current and of the electrostatic forces is negligible compared with that of the current of conduction, but yet so rapidly alternating that the currents do not penetrate into metallic conductors further than a thin outer skin. In such a case the magnetic induction must be, close to the surface of a conductor, wholly tangential,—being continuous normally and null inside. As the magnetic field arises from the conduction currents, its modification by the displacement currents in the surrounding dielectric is negligible: thus by Ampère's circuital relation the magnetic force (α, β, γ) in the dielectric, where there is no sensible current, is derived from a magnetic potential U. The equations of propagation of magnetic force in the surrounding space are then equivalent to the equation of propagation of aereal sound-waves in which U represents the condensation, the velocity of propagation being however that of radiation instead of that of sound; indeed with these restrictions, nothing is gained in accuracy by not taking the propagation to be instantaneous. The condition of magnetic force tangential along a conductor makes the conductor correspond to a solid wall. Hence there is perfect formal analogy between the maintained magnetic oscillations of long wave-length in the space between the conductors, and standing aereal sound-

waves in a region of the same form. Thus the acoustical analysis for standing waves long compared with the dimensions of the boundary has an application to alternating electric fields. Moreover the conductors in the alternating electrodynamic field are repelled from the alternating source (by Maxwell's theorem above) with the forces arising from a normal pressure of intensity $\mathfrak{H}^2/8\pi$: this again is an exact analogue, except as to sign, of the *attraction* of solid obstacles by the aereal sound-waves issuing from a vibrator. With wave-lengths long, as here, compared with the dimensions of the obstacles, the acoustic problem is virtually one of oscillatory flow of a fluid not sensibly subjected to compression: and the pull on the wall is simply the defect of pressure due to the head of velocity at the place. Whenever the electric oscillation is of a steady character we may express the corresponding result in Faraday's manner by saying that a small piece of metal, say copper or iron, is urged towards the places where the energy of the steady oscillation is weakest, the force so urging it depending only on its form and not on its material. A small elongated piece of metal will moreover tend to set itself across the lines of alternating magnetic force for the same reason that a small elongated obstacle will tend to set itself across lines of aereal flow. It might thus prove practicable to use copper filings (suitably treated to avoid cohesion) to map out the magnetic field of an alternating motor, just as iron filings are employed to map out a steady magnetic field: each filing will tend to set its length *across* the magnetic field, not along it, and will at the same time tend to move towards regions of weaker alternating magnetic force.

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The penetration and transmission of progressive dielectric waves, such as Hertzian aethereal waves, through metallic pipes and channels is however not (or is only roughly) analogous to the collection and transmission of aereal sound-waves of the same length through speaking tubes. Thus there is only rough general similarity between electric telegraphy across space and the corresponding sound signals. The action of the ground and of intervening obstacles is practically much the same in both cases, though non-metallic obstacles would usually be more

absorptive in the electric case. In both cases the radiation attenuates rapidly with increasing distance from the source, roughly according to the same law. Situations which are suitable for the one are also suitable for the other. The advantage possessed by the electric method, irrespective of greater initial intensity and greater delicacy in the receiver, is facility in picking up the signal from the surrounding space: the function of the wire, extending up into the air, that is usually connected with the receiver is presumably to surmount intervening obstacles, or to tap a stronger stratum of radiation wherever it happens to be: for the different parts of its length could hardly have time to act in a cumulative manner, unless the waves are very long**.

Mechanical Pressure of Radiation

82. In cases in which radiation is important, the Maxwellian electrodynamic stress-formula is, as seen above, inapplicable. The case in which it comes nearest to being useful for obtaining exact results is that of electric disturbance, of any kind, in a uniform non-conducting isotropic medium which is at rest and also devoid of magnetization. In that case the electrokinetic part of the mechanical bodily forcive (X, Y, Z) is of type

$$X = (v - \dot{g})\gamma - (w - \dot{h})\beta$$
$$= (1 - K^{-1})(v\gamma - w\beta),$$

because the total current is K times the aethereal current; so that the forcive acting over any region of such a uniform medium is statically equivalent to $1 - K^{-1}$ times Maxwell's traction over the outer boundary.

83. The discussion of the mechanical forcives connected with the absorption and reflexion of radiation is thus best conducted directly. For the case of a beam of light passing across a medium whose properties change but slightly in lengths comparable with a wave, there is practically no reflexion; and it will appear that it is only absorption that is accompanied by mechanical force, which is in the direction of propagation of the light and depends on the rate of absorption of energy. But at an interface where the transition is practically abrupt,

** The wire connected with the radiator may however secure this.

there will also be a turning back of energy of the waves owing to a finite reflexion, and this will involve mechanical traction acting on the interface. The validity of Fresnel's laws of reflexion at an interface of transparent optical media shows that, even in comparison with the length of light-waves, the layer of transition is thin: as regards Hertzian waves it is of course obviously so. It is natural to assume like abruptness in the transitions between absorbing optical media, where the data are as yet hardly precise enough to test the fact.

Consider the simple case of a train of plane-polarized waves advancing in the direction of x, so that all the quantities are functions of x only, the electric force being $(0, Q, 0)$ and the magnetic force $(0, 0, \gamma)$. We shall retain a magnetic permeability μ in view of possible application to long Hertzian waves: and we shall suppose (so far as a simple analysis is found to permit) that both μ and K and also the conductivity σ are functions of x. The equations of propagation are

$$4\pi v = -\frac{d\gamma}{dx}, \quad \frac{dQ}{dx} = -\mu\frac{d\gamma}{dt}, \quad v = \sigma Q + \frac{K}{4\pi C^2}\frac{dQ}{dt}.$$

The mechanical force per unit volume is given by equations of type

$$X = \left(v - \frac{dg}{dt}\right)\gamma - \left(w - \frac{dh}{dt}\right)\beta + f'\frac{dP}{dx} + g'\frac{dP}{dy} + h'\frac{dP}{dz};$$ cf. p105

thus here $$X = \left(v - \frac{dg}{dt}\right)\gamma, \quad Y = 0, \quad Z = 0.$$

Now $$\int_{x_1}^{x_2} v\gamma\, dx = -\left|\gamma^2/8\pi\right|_{x_1}^{x_2},$$

so long as γ is continuous between the limits of integration, which is always. Also

$$-\int \gamma\frac{dg}{dt}\,dx = -(4\pi C^2)^{-1}\int \gamma\frac{dQ}{dt}\,dx;$$

while for harmonic oscillatory motion of period $2\pi/n$

$$-n^2\gamma = \frac{d^2\gamma}{dt^2} = -\frac{1}{\mu}\frac{d^2Q}{dx\,dt};$$

hence this integral is equal to

$$-(4\pi C^2)^{-1}\int \frac{1}{\mu n^2}\frac{dQ}{dt}\frac{d^2Q}{dx\,dt}dx,$$

that is,

$$-(8\pi C^2\mu n^2)^{-1}\left|\left(\frac{dQ}{dt}\right)^2\right|_{x_1}^{x_2},$$

provided that μ is constant, and also that dQ/dt is continuous throughout the range of integration as is always the case, though K may change gradually or abruptly.

We have thus

$$\int_{x_1}^{x_2} X dx = -\left|\frac{\gamma^2}{8\pi} + \frac{1}{8\pi C^2\mu n^2}\left(\frac{dQ}{dt}\right)^2\right|_{x_1}^{x_2},$$

which gives the aggregate mechanical forcive on the stretch of the medium between x_1 and x_2 in the form of pressures, of the amount in $|\ldots|$, acting on its ends. Thus for the simple harmonic time-alternations of γ and Q that we have assumed, the time-average of the pressure on either end is

$$(16\pi)^{-1}(\gamma^2 + Q^2/\mu C^2),$$

γ and Q now representing the magnetic and electric amplitudes of the vibration; this is the sum of the mean kinetic and potential energies per unit volume of the radiation, less that involved in the electric polarization of the molecules, divided by μ; hence on any portion of the medium there is a mechanical force, directed along the waves, equal per unit cross-section to the difference of these densities of energy at its ends.

In a transparent medium

$$\left(\frac{dQ}{dt}\right)^2 = \frac{C^2}{K\mu}\left(\frac{dQ}{dx}\right)^2 = \frac{C^2\mu}{K}\left(\frac{d\gamma}{dt}\right)^2;$$

so that the above internal pressure may be expressed in the form

$$(8\pi)^{-1}\left\{\gamma^2 + \frac{1}{Kn^2}\left(\frac{d\gamma}{dt}\right)^2\right\}.$$

If there is in the medium a directly incident wave whose vibration at the interface is $\gamma_1 \cos nt$ and also a reflected wave $\gamma_2 \cos(nt-\epsilon)$, and a refracted wave, this result may be applied to a layer of the medium containing the interface; thus there

will be a mechanical traction on the interface represented by a difference of pressures on its two sides, that on the incident side being

$$(8\pi)^{-1}[\{\gamma_1 \cos nt + \gamma_2 \cos(nt-\epsilon)\}^2 + K^{-1}\{\gamma_1 \sin nt + \gamma_2 \sin(nt-\epsilon)\}^2].$$

In air or vacuum that is $(\gamma_1^2 + \gamma_2^2 + 2\gamma_1\gamma_2 \cos\epsilon)/8\pi$, or simply $\gamma^2/8\pi$, where γ is the amplitude of the resultant magnetic vibration on that side.

When the radiation is directly incident on an opaque medium γ and dQ/dt are null in its interior: so that, when the surrounding medium is air or vacuum, its surface sustains in all a mechanical inward normal traction of intensity $\gamma^2/8\pi$, that is, equal to the mean energy per unit volume of the radiation just outside it, in agreement with Maxwell's original statement.

Absorption: Perfectly Black Bodies: Perfect Reflectors

84. On the molecular conception of the constitution of a material body capable of propagating radiation, which is the foundation of the present theory of material media, it becomes important to inquire whether there can exist either a perfectly absorbent body or a perfect reflector, in view of the general theory of exchanges of radiation that is usually based, after Balfour Stewart and Kirchhoff, on arguments which assume the theoretical existence of such bodies.

The following analysis, forming in the main a scrutiny of the nature of steady absorption of radiation, will determine how closely either of these ideal properties may be approximated to, in cases in which the transition at the surface is so abrupt that the ordinary dynamical laws of reflexion and transmission apply.

Consider a train of plane waves travelling parallel to the axis of x in an absorbing medium, the waves being polarized so that the magnetic force is γ, parallel to the axis of z, and the electric force is Q, parallel to the axis of y. The equations of propagation are, as above,

$$4\pi v = -\frac{d\gamma}{dx}, \quad \frac{dQ}{dx} = -\frac{dc}{dt},$$

$$v = \sigma Q + \frac{K}{4\pi C^2}\frac{dQ}{dt},$$

in which $c = \mu\gamma$, the coefficient μ being sensibly different from unity only in magnetic substances and then only for long waves.

It follows that

$$\frac{d}{dt}(4\pi\mu v) = \frac{d^2Q}{dx^2},$$

and therefore

$$\sigma\frac{dQ}{dt} + \frac{K}{4\pi C^2}\frac{d^2Q}{dt^2} = \frac{1}{4\pi\mu}\frac{d^2Q}{dx^2},$$

which is the equation of propagation.

Considering the case of radiation of period $2\pi/n$ so that

$$Q = Q_0 e^{\iota nt - px},$$

this gives $$\sigma n\iota - \frac{K}{4\pi C^2}n^2 = \frac{1}{4\pi\mu}p^2,$$

or $$p = \frac{(K\mu)^{\frac{1}{2}}}{C}n\iota\left(1 - \frac{4\pi\sigma}{nK}C^2\iota\right)^{\frac{1}{2}}, \text{ say } = p_1 + \iota p_2.$$

On separating the real parts, we have

$$Q = Q_0 e^{-p_1x}\cos(nt - p_2x),$$

corresponding to

$$c = Q_0 n^{-1}(p_1^2 + p_2^2)^{\frac{1}{2}} e^{-p_1x}\sin(nt - p_2x + \epsilon), \text{ where } \tan\epsilon = p_2/p_1,$$

and $$v = Q_0(4\pi n\mu)^{-1}(p_1^2 + p_2^2)e^{-p_1x}\sin(nt - p_2x + 2\epsilon).$$

Thus the magnetic flux is in a different phase from the electric force, involving a diminution in their vector product which determines the energy transmitted across any plane; while the lag of phase of the electric current behind the time-gradient of the electric force is twice as great as that of the magnetic flux. The energy per unit volume of the radiation at any part of the wave consists of an electric part $\frac{1}{2}Qg''$, or $KQ^2/8\pi C^2$, and a magnetic part $\mu\gamma^2/8\pi$: the ratio of the time-averages of these parts is

n^2/c^2

$$K\mu\cancel{C^4/n^2}(p_1^2 + p_2^2), \text{ or } (1 + 16\pi^2\sigma^2C^4/n^2K^2)^{-\frac{1}{2}},$$

which is constant, but not unity except for transparent media. The time-rate of propagation of radiant energy is, by Poynting's theorem, which applies since the matter is at rest,

$$\frac{dE}{dt} = \frac{Q\gamma}{4\pi} = \frac{Q_0^2}{4\pi n\mu} e^{-2p_1x} \left(\tfrac{1}{2}p_2 + \text{periodic terms}\right).$$

Across the plane $x = 0$ it is therefore on the average $Q_0^2 p_2/8\pi n\mu$, which corresponds to a density of energy equal to mean square of electric force divided by $4\pi\mu_{K}$ travelling (n/μ) with the speed n/p_2 of the waves. This involves the result that only the fraction

$$\frac{2}{\cancel{K\mu}} \Big/ \left\{1 + \left(1 + \frac{16\pi^2\sigma^2 C^4}{n^2 K^2}\right)^{\frac{1}{2}}\right\}$$

of the total energy of the wave-system can be considered as propagated; in the case of an undamped wave-train this is ~~only~~ the whole of it ~~purely aethereal part~~. The aethereal wave-train, passing across the material medium, sets its molecules into sympathetic independent electric vibration: the energy of these vibrations constitutes a part, of the total energy per unit volume, but that part is not all propagated. This remark applies See Ca equally to all optical theories in which change of velocity of propagation is traced to the influence of sympathetic vibrations of the molecules; in fact it applies to all cases in which velocity depends upon wave-length. Delicate considerations then arise as to the manner in which the front of a train of simple waves advances in a material medium: a simple wave-train with an abrupt front could not carry sufficient energy to establish itself as it goes along, and in fact the complete energy of the train is only established with the smaller velocity of the group of waves which constitutes the real advancing disturbance*.

85. To determine how perfect the absorption of an actual medium with an abrupt interface may become, we have to solve the problem of direct reflexion. Let the expressions for the incident, reflected, and transmitted beams be respectively

$$Q = \exp \iota\left(nt - \frac{n}{C}x\right), \quad \gamma = \frac{1}{C}Q$$

$$Q' = B\exp \iota\left(nt + \frac{n}{C}x\right), \quad \gamma' = -\frac{1}{C}Q'$$

$$Q_1 = C\exp(\iota nt - px), \quad \gamma_1 = -\frac{\iota p}{\mu n}Q_1;$$

* Cf. Rayleigh, 'Theory of Sound,' vol. i, Appendix, referring to O. Reynolds.

where by the equation of propagation

$$p^2 = -\frac{K\mu}{C^2} n^2 \left(1 - \frac{4\pi\sigma}{K}\frac{C^2}{n}\iota\right),$$

so that for downward waves

$$p = +\mu n C^{-1} \lambda (\sin \tfrac{1}{2}\theta + \iota \cos \tfrac{1}{2}\theta),$$

where $\tan\theta = \frac{4\pi\sigma}{K}\frac{C^2}{n}$, and $\lambda = \left(\frac{K}{\mu\cos\theta}\right)^{\frac{1}{2}}$.

A real solution will ultimately emerge on rejection of the imaginary parts of these expressions.

At the interface $Q + Q' = Q_1$, $\gamma + \gamma' = \gamma_1$,

so that $$1 + B = C, \quad 1 - B = -\iota \frac{pC}{\mu n} C:$$

hence
$$\begin{aligned} \tfrac{1}{2}C &= \left(1 - \frac{pC}{\mu n}\iota\right)^{-1} \\ &= \frac{1 + \lambda\cos\frac{1}{2}\theta + \iota\lambda\sin\frac{1}{2}\theta}{1 + 2\lambda\cos\frac{1}{2}\theta + \lambda^2}; \end{aligned}$$

and
$$\begin{aligned} B &= \left(1 + \frac{pC}{\mu n}\iota\right) \Big/ \left(1 - \frac{pC}{\mu n}\iota\right) \\ &= \frac{1 - \lambda^2 + 2\iota\lambda\sin\frac{1}{2}\theta}{1 + 2\lambda\cos\frac{1}{2}\theta + \lambda^2}. \end{aligned}$$

The amplitude of the reflected vibration is the modulus of B, given by

$$|B| = \frac{(1 - 2\lambda^2\cos\theta + \lambda^4)^{\frac{1}{2}}}{1 - 2\lambda\cos\frac{1}{2}\theta + \lambda^2}.$$

Thus $|B|$ cannot vanish, so that there cannot be a perfect absorber or "absolutely black body" bounded by an abrupt interface; but that does not preclude the possibility of a molecular constitution of the interface, of a loose and gradual kind such as may exist in lamp-black, which might absorb light as soft porous bodies absorb sound*. It may readily be shown that the body is a total reflector, $|B|$ being unity, only when K or σ is infinite, or else when ~~θ is equal to $\frac{1}{2}\pi$ and therefore σ/K is infinite.~~ both of them vanish

* The character of this latter process, as depending on degradation through heat-conduction in the pores, has been fully elucidated by Rayleigh, 'Theory of Sound,' Ed. 2 §§ 350—1.

Relation of Intensity of Radiation to Temperature

86. An important application has been made by Boltzmann†, following Bartoli, of the result above verified, that the pressure exerted on an opaque body by direct radiation in free space is equal to the density of radiant energy just in front of it.

If we consider an enclosure in which a steady state of radiation is established so that radiation is traversing it equally in all directions, we have to average the direct pressures for the different values of the angle of incidence θ, there being no sideway pressure: the foreshortening of the element of area pressed gives a factor $\cos\theta$, and resolving the pressure normally gives another $\cos\theta$, so that the resultant pressure on the interface is equal to one-third of the total density of radiant energy in the enclosure.

Now let us consider an opaque body (it need not be perfectly black), surrounded by a space filled with the radiation appropriate to its temperature, which is bounded externally by a totally reflecting and therefore impervious flexible envelope. This envelope sustains the pressure of the radiation inside it: by changing its form the volume enclosed can be contracted, and the radiant energy that filled the part of the volume that thus disappears will be driven into the central body. But the total energy that so disappears is this volume-distribution of vibrational energy together with the work done by the envelope against the radiant repulsion, amounting in all to four times the latter part. This process is reversible; so that the energy emitted in radiation is in part mechanically available. But it is not to be classed with mechanical energy, because the availability is not complete; it is not possible to execute a cyclic process by which radiant energy is changed directly into mechanical energy.

87. We can however construct a system involving differences of temperature through which a reversible cycle can be operated, and to which therefore Carnot's principle can be applied. We have to suppose an interior body A_1 at tempera-

† Cf. Rayleigh, *Phil. Mag.* 1898.

ture T_1, surrounded by an exterior body A_2 at temperature T_2, but separated from it by a perfectly reflecting shell in the space between, which will prevent equalization of temperature through passage of radiation from the one body to the other. The spaces on the two sides of the shell will each be filled with radiation of the constitution and density corresponding to the temperature of the body on that side. We can imagine an ideal pump, constructed of perfectly reflecting material, that will pump radiation from the one side of this shell to the other, working against the difference of radiant pressure between the two sides: when the piston of such a pump is drawn out, the energy of the radiation that is isolated in the cylinder must be diminished by the work done by its pressure on the retreating piston. The result will be that if p_1 and p_2 are the pressures of radiation on the two sides, then for each unit volume of radiation transferred by the pump from outside to inside, the outer body A_2 must emit energy of amount $4p_2$, made up of the energy E_2 of the radiation and the work p_2 done by it on the piston, while the inner must absorb exactly what remains of this after the mechanical work W is performed. Now by Carnot's principle, we have for such an engine working reversibly between temperatures T_2 and T_1

$$\frac{H_2}{T_2}=\frac{H_1}{T_1}, \quad =\frac{W}{T_2-T_1}.$$

In the present case, if the temperatures T_2 and T_1 on the two sides of the partition differ by a finite amount, the determination of the work W will involve an integration: let us therefore take the difference of temperatures to be infinitesimal**, say δT, when the work will be equal to $p_2 - p_1$, or δp, to the first order. As H_2 is $4p_2$ or $\frac{4}{3}E_2$, we have thus

$$\frac{\frac{4}{3}E_2}{T_2}=\frac{\frac{1}{3}\delta E}{\delta T},$$

which yields on integration

$$\log E = 4 \log T + \text{const.}$$

Thus we arrive at the empirical law enunciated by Stefan, that the density of radiant energy corresponding to any given

** See § 87** at the end of this Chapter.

absolute temperature is proportional to the fourth power of that temperature*. A consideration of the further developments of W. Wien†, who takes into account the Doppler effect in order to obtain a relation between the constitutions of the complete radiations at different temperatures, would take us too far from our subject.

The argument here sketched implies that absolutely none of the radiation is absorbed by the surface layers of the dividing shell or by the pump; for such absorption would soon generate an appreciable change of temperature in this surface layer and so modify the application of the thermodynamic principle. This requires the screen to be of a purely ideal kind, in that it is absolutely totally reflecting for all radiation, a property which cannot be possessed by any actual screen of molecular material constitution. The question arises whether this introduction of an ideal mechanism vitiates the thermodynamic proof: as its function is only to produce constraint in bulk, not as regards individual molecules, and therefore is a mechanical one, it can reasonably be held that its use is legitimate. In fact if the radiation is constituted of Hertzian waves of considerable length, a metallic screen of good conducting quality approximates very closely to the ideal required, as it produces practically complete reflexion and therefore no absorption. Another assumption in this mode of argument is that the motion of the screens does not disturb the structure of anything that may exist in the space between the black bodies: thus that space must be empty of all matter. We must also consider the screens to be freely pervious to aether, which may form a real difficulty in the argument. We have seen moreover that in a material dielectric medium the pressure of a train of radiation directly incident on a perfect reflector or on a black body is not equal to the density of the radiant energy just in front of it. See pp. 145–148.

* With the above may be compared the thermodynamic argument which shows that in weakly paramagnetic material the magnetic susceptibility varies inversely as the absolute temperature: *Phil. Trans.* 1897 A, p. 287.

† *Berlin. Sitzungsberichte*, 1893.

On Dynamical and Material Symmetry: General Deductions

88. The criterion that the motion of a system may be reversible is that on reversing the time, that is on writing $-t$ in place of $+t$, the dynamical equations shall remain unchanged: for this analytical operation involves the change of sign of every velocity such as dx/dt, while the coordinates of position such as x are not changed: nor are any of the accelerations such as d^2x/dt^2 thereby changed. A non-dissipative dynamical system is reversible provided the kinetic energy T is a function of the velocities in which all the terms are of the second or other even degree with coefficients involving the coordinates in any manner, while the potential energy W is a function of the coordinates alone: for change of sign of t then leaves unaltered the Lagrangian function $T-W$ on which the course of the motion depends. But if the dynamical problem has been modified after the manner of Routh and Lord Kelvin, by eliminating such of the coordinates as appear only through their velocities in the expression for the energy, and introducing in place of them the corresponding momenta, which are under these circumstances constant, then the modified Lagrangian function contains mixed terms, involving these constant momenta and the remaining velocities each in the first degree, in addition to quadratic terms in the remaining velocities and in the cyclic momenta respectively: and the motion thus specified cannot be reversed unless these cyclic constant momenta are reversed at the same time. If the motion of any given system prove to be reversible, there can be no latent cyclic momenta involved in it: there may be latent possibilities of cyclic motion, that is coordinates representing cyclic freedom, but the momenta attached to them must then be null. As the coordinates cannot in the nature of the case be all cyclic, the only kind of exception to this irreversible quality of dynamical systems involving latent cyclic momenta is the approximate one in which the part of the kinetic energy involving in any way the remaining velocities is negligible: this will be the case when the remaining sensible coordinates of the system change their values very slowly; and the system may then be described as a static system of cyclic character. The

changes of the system, and its configurations of steady cyclic motion, will then be determinated solely by a modified static energy function, in the same manner as the equilibrium configurations and the trend of all very slow changes of an ordinary acyclic system are determined by its potential energy alone. But in this the only case of reversibility of a material system with latent cyclic momenta, the property is only approximate: it disappears altogether when the velocities of the sensible coordinates become comparable with those of the latent ones.

89. The general dynamical system involving latent cyclic momenta, as well as finite sensible momenta, has been utilized* as an analytical representation of the thermodynamic relations of an ordinary material system. The main feature of absence of direct reversibility is common to both: thus in both cases the sensible velocities enter in the first degree into the available energy function. But the range of the analogy is a restricted one: in the thermal system there can be no really steady motion until all relative sensible motions have disappeared and it moves as a rigid body, while this is clearly not the case for every cyclic system: the latter is therefore in one respect more general. In the former system the thermal energy gives rise to forces (passive reactions) which uniformly act against and retard the motions belonging to the sensible coordinates, being in part energy in the individual molecules of a more or less cyclic type, but also in part energy of their irregular translatory velocities. The latter part of it is certainly related to forces that are wholly viscous: it must therefore be excluded from a purely cyclic scheme, the analogy of which is accordingly confined to systems not subject to transmission of heat by conduction between bodies at finitely different temperatures**.

90. There is another simple transformation, analogous to reversal, which can sometimes serve as a clue to the dynamical relations of physical systems. If the sign of the x coordinate of position of each point of the system is changed, but the sign of the time not changed, the actual motion is changed into its

* Cf. von Helmholtz's memoirs on the Dynamics of Monocyclic Systems, in Vol. iii of his Collected Papers.

** A closer relation may possibly be developed between cyclic systems and the constitutive molecular part of the energy.

reflexion in a mirror lying along the plane of yz*. Right-handed relations with reference to the axis of x are thereby changed to left-handed ones. If in any given case it is known that the reflected motion can exist spontaneously, just as the original motion, it follows that there is nothing right-handed or left-handed, nothing chiral in Lord Kelvin's phrase, with regard to this axis, in the constitution of the system. For example, consider the propagation of a right-handed circularly polarized train of light-waves in a sugar-solution: its velocity usually differs slightly from that of a left-handed train, thus giving rise to the phenomena of rotation of the plane of polarization: on reflexion it becomes a left-handed train travelling backwards, and goes with a different speed, that appropriate to left-handed waves: thus the medium has chiral properties and change of sign of x must affect the equations of propagation. But if the wave-train is travelling in a medium magnetized along the direction of x, the change of its chirality by reflexion no longer produces any change in the velocity: there is thus no essential chirality in the medium, and the magnetic rotatory polarization must be traced to some other source. It is in fact related to the imposed magnetism, which is a vector agency directed along the axis of x, and therefore can be connected with difference of velocity for different directions along that axis.

91. When the reflexion is direct, the reflected motion is simply the reversed motion with chirality changed: thus in a simply chiral medium the motion is completely reversible; but in the magnetized medium it is not so, and the complete condition of reversion must then involve the reversal of the magnetic field or other extraneous vector agency which causes the rotation by interacting with the material system. It will be of interest,

* This case of reflexion in the plane of yz is the most general one that need be considered here. For successive reflexions in two planes produce merely a bodily rotation, round their line of intersection as axis, and of amount equal to twice the angle between them, on the principle of the sextant: thus any even number of plane reflexions produces a simple bodily displacement, and any odd number produces a bodily displacement combined with a single reflexion. The latter reduction is however not unique: the reflexion can be changed into another reflexion by adding on a rotation round the intersection of their planes, and this may be chosen so as to simplify the bodily displacement.

following Lord Kelvin's train of ideas, to examine how much more closely we can specify the character of an imposed physical vector agency that will be competent to produce optical rotation. Dynamically, an agency which is to affect the character of the rotational motion of a circularly polarized wave must be of the nature of a moment round the axis rather than a translatory effect along it—it must in fact be a moment of momentum rather than a linear momentum—the only case of exception being when the medium has itself chiral property, in which case a linear momentum reacting with that property would produce an influence of this kind, but of the second order. The reason of this restriction is simply that if an imposed agency along an axis is to affect a motion around it, the analytical expression of the relation between them must involve screw coefficients with respect to the medium: if the medium has no intrinsic screw relation in its constitution the effect must be null*. Thus an imposed magnetic field must partake dynamically of the nature of rotation round its axis: the only way in which rotation of the aether itself has been imagined for the purpose is by the assumption (by Maxwell†) of vortical whirls in it such as could only be associated with contained matter. This, conjoined with the fact that magnetic rotation does not exist in free aether, goes far to establish that an imposed magnetic field implies internal rotation in the molecules of the matter with respect to its axis, which agrees with our modified Weberian theory ascribing magnetism to the orientation of the molecular orbits of the electrons associated with the molecules. We can in fact, on the assumption that the molecule is constituted solely of moving electrons with or without added extraneous inertia, define the magnetisation per unit volume as proportional to the moment of momentum per unit volume of the internal molecular motions, *provided that* in estimating it

* In a similar manner it would appear that an extraneous modification of a medium, constituted by an arrangement of fluid vortices in planes at right angles to the axis, could only produce optical rotation by modifying a structural chiral property already existing in the medium.

† 'Treatise' ii § 822 'On the hypothesis of molecular vortices,' the title suggesting that the interpretation above given was present to Maxwell's mind.

regard is had to the signs as well as the velocities of the electrons: this latter restriction supplying the reason why a magnet does not exhibit gyrostatic mechanical reactions**.

92. *Modification of the physical constants of a material medium by translatory motion through the aether.*—An important application of these principles of symmetry arises in determining to what extent the dielectric and magnetic constants of a material medium are affected by motion of translation through the aether. Let us suppose that the molecular structure of the medium is polar, as in the ordinary theory of magnetic and dielectric media, and that it is devoid of chiral quality. Then the only change produced by reversal of its velocity of translation is that this velocity has now the same relations to the negative poles that it previously had to the positive ones. Now on any view hitherto conceivable of electric and magnetic polarity, based on a stationary aether, a motion of translation of the medium must affect and be affected by a polarity in the same way as by the reversed polarity. Hence reversal of the velocity of translation will not affect anything essential: the influence of the translation therefore depends on even powers of its velocity compared with that of radiation, or—to an approximation sufficient for all purposes—it is proportional to the square of the ratio of the velocity of translation of the medium to the velocity of radiation. The changes in the physical constants, including therein possible changes of dimensions, of the material medium produced by translation through the aether are therefore second-order effects.

** It seems worthy of notice that Lord Kelvin's argument, connecting magneto-optic rotation with rotatory motion of matter in the magnetic field, rests essentially on the same foundation as Newton's statement (*Principia*, Scholium to *Definitiones*) that the amount of the absolute rotation of a material system may be detected by the centrifugal reaction it affords to circular motion, for example by the change of form of the steady parabolic surface of liquid in a rotating bucket when the direction of the rotation is reversed. In the Newtonian experiment it is implied that the forces between the parts of the system depend on configuration only: the structure of the system may thus be chiral, but it must not involve any rotational affection relative to a definite axis or direction in it, which would impart gyrostatic quality to its inertia.

But there is one class of possible exceptions to this result, that namely of directed or chiral properties as distinguished from bipolar ones. There is nothing in the mere formal character of the quantities involved to prevent the optical rotation of vector type arising from an imposed magnetic field from being affected to the first order by a motion of translation of the material medium. The coefficient of the chiral optical rotation in a chirally constituted medium will not however be affected to the first order, because reversal of the medium end to end will not reverse its chiral relation: but in such a medium there is room formally for a new rotatory effect, arising from convection, of the same type as magnetic rotation, and of the order of the product of its chiral coefficient and the first power of the ratio of its velocity of convection to the velocity of radiation.

On our present theory of a stagnant aether and discrete distribution of electricity both these possible effects should certainly vanish up to the first order (§ 109): while the second-order theory to which that view leads (§ 112), which satisfies the requirements of the Michelson interference experiment, would also require the effect of convection on the rotatory property to be null up to the second order. Cf. §§ 141—3.

87**. The exact scope of the relations of § 87 will appear more clearly in a procedure involving finite range of temperature, where we follow in the main a process already given by Boltzmann*. Let E denote the energy per unit volume of the steady distribution of radiation in an enclosure, and p the pressure it exerts on the walls. When the walls of the enclosure are perfectly reflecting and therefore adiabatic, there will be no connexion between E and the temperature of the walls: but in all cases p must be a function of E alone, say $f(E)$. When the volume v of such an adiabatic enclosure is altered, by change of its form or by advance of a piston, the conservation of energy in its interior requires

$$d(Ev) = -p\,dv,$$

* *Wied. Ann.* xxii. 1884, p. 294.

so that $$vdE = -\{E + f(E)\}\,dv, \quad \text{.................(i)}$$

which is the adiabatic relation between the intensity of the radiation and the volume of the enclosure containing it.

We can now consider volume v_1 of radiation admitted into the cylinder of the ideal pump of § 87, at the intensity E_1 of the radiation surrounding the body of lower temperature T_1, then compressed adiabatically by the piston to a volume v_2 at the intensity E_2 which is in equilibrium with the other body of higher temperature T_2. This radiation can thus be considered as initially emitted by the first body, and finally absorbed into the second body, but in increased ~~diminished~~ amount because the energy required for the performance of this mechanical work of compression must be added ~~deducted~~. The process is mechanically reversible; so that the application of Carnot's principle would give

$$v_1 \frac{E_1 + p_1}{T_1} = v_2 \frac{E_2 + p_2}{T_2} \text{..................(ii)}$$

As the variables E_2, p_2, v_2 refer to any state that can be derived from the state E_1, p_1, v_1 by an adiabatic process for which (i) holds, we have, on omitting the suffixes, the general relations

$$vdE = -(E + p)\,dv,$$

$$v\frac{E + p}{T} = A,$$

where A is a constant independent of T and v. To eliminate v, and thus obtain the relation between E and p, we take differentials of the second of these equations, giving

$$vd\frac{E + p}{T} = -\frac{E + p}{T}\,dv:$$

hence from the first equation

$$d\frac{E + p}{T} = \frac{dE}{T},$$

so that $$\frac{dp}{E + p} = \cancel{-}\frac{dT}{T} \text{.....................(iii)}$$

The adiabatic relation (i) between the volume v and the mechanical pressure p of radiation may also be expressed in the form

$$v(E+p)=AT, \text{ or } v\frac{dp}{dT}=A. \quad \ldots\ldots\ldots\ldots\text{(iv)}$$

The existence of the mechanical pressure p determined by (iii) would thus render cyclic processes involving radiation consistent with Carnot's principle.

According to the investigation of §§ 83, 86, $p=\frac{1}{3}E$, *provided* the radiation is in free space, not in a material medium. By (iii) this gives E proportional to T^4. By (iv) it gives for adiabatic change of volume of radiation the relation $pv^{\frac{4}{3}}$ constant.

In estimating the mechanical value or availability of a distribution of radiant energy existing in free aether, we must thus assign to each portion of it a temperature equal to that of the walls of the enclosure that would be in equilibrium with it. But this principle applies only to the steady uncoordinated radiant energy in an enclosure, for which there is no particular direction of propagation, or which may be considered as propagating itself equally backwards and forwards in all directions in the enclosure. The radiation travelling in a definite direction from a radiant source can on the other hand theoretically be restored to its original density by aid of a lens or reflector; its theoretical availability therefore remains unimpaired as it becomes less intense with increase of distance from the source. By well-known optical theorems, it cannot be concentrated to greater intensity than it had originally: that would in fact involve *increase* of availability without compensatory decrease elsewhere, in other words perpetual motion.

The difficulty mentioned on p. 139 as to the possibility of imagining an ideal screen impervious to radiation but pervious to the aether is not peculiar to the present discussion; for precisely the same properties are required in an adiabatic envelope in the ordinary applications of Carnot's principle. What is wanted to establish the argument on a practical basis is a first approximation to this impermeable quality: in

theoretical physics it is a common procedure to idealize from the imperfect qualities of actual matter to a limiting perfection. As has been already remarked, if the radiation is of the type of long Hertzian waves, a metallic screen possesses the requisite properties: it is therefore perhaps legitimate to imagine a screen of ideal very fine-grained matter which would serve the purpose for the much shorter waves of light.

SECTION III

CHAPTER IX

INFLUENCE OF STEADY MOTION ON AN ELECTROSTATIC MATERIAL SYSTEM

93. THE general equations formulated in the preceding section enable us to treat in detail the question whether there is any change in the steady distribution of electric charges on a system of conductors, when they are set in motion, whether along with dielectric bodies or not, through the aether. In order that a steady electric state may be possible, without permanent currents of conduction, the configuration of the matter must remain unchanged; moreover it must always present the same aspect relative to its motion, and also relative to extraneous electric and magnetic fields when such are present. The motion of the material system must therefore be a uniform spiral or screw motion on a definite axis fixed in the aether, including as special cases translation along and rotation round this axis: and the imposed or extraneous fields when such exist must be symmetrical round this axis. Otherwise the circumstances would be continually changing, and there could be no steady state of electrification of the system.

In such steady state, when it exists, the magnetic induction through every circuit moving along with the material system remains constant on account of the steadiness. It follows by the Faraday circuital relation, which holds good universally for circuits moving with the matter, that the line integral of the electric force (P, Q, R) round every circuit

vanishes. Hence the electric force is derived throughout the field from an electric potential V, so that

$$(P, Q, R) = -(d/dx,\ d/dy,\ d/dz)\, V.$$

Moreover, inside the conductors the electric force must vanish, for otherwise electric separation would be continually going on, leading to steady currents of conduction; thus in problems in which such currents are by symmetry excluded on account of the absence of a return path, the potential V must be constant throughout each conductor and therefore over its surface.

The aethereal force (P', Q', R') which produces elastic displacement (f, g, h), equal to $(4\pi C^2)^{-1}(P', Q', R')$, in the aether is connected with this electric force (P, Q, R), which acts on the electrons and thus produces movement of electrification*, by the relation

$$(P', Q', R') = (P - qc + rb,\ Q - ra + pc,\ R - pb + qa),$$

where (p, q, r) is the velocity of the matter.

94. For purposes of analysis it is clearly convenient to refer the problem of steady motion to a space, or say an ideal frame, moving along with the material system; thus (p, q, r) is the velocity of this moving space at the point (x, y, z), with reference to the stagnant aether.

When the region surrounding the material conductors is free space, the total current in it is the displacement current in the aether, equal to $d/dt\,(f, g, h)$ where the differentiation refers to a point fixed in the stagnant aether. Now the condition of steadiness of state relative to the moving axes gives $\delta/dt\,(f, g, h) = 0$ where δ/dt represents

$$d/dt + p\,d/dx + q\,d/dy + r\,d/dz.$$

Hence the total current at a point (x, y, z) in free aether is

$$-\left(p\frac{d}{dx} + q\frac{d}{dy} + r\frac{d}{dz}\right)(f,\ g,\ h).$$

But in dielectric insulating matter there is to be added to this aethereal current the effect of the convection of the steady

* The electric force at a point in the free aether is here the force that would act on unit charge situated at the point and moving with the material system.

material polarization (f', g', h'), which (§ 63) is represented for purposes of continuous electromotive analysis by a current

$$\left(p\frac{d}{dx} + q\frac{d}{dy} + r\frac{d}{dz}\right)(f', g', h').$$

Case of uniform translation

95. When the motion of the material system is restricted to one of uniform translation so that (p, q, r) is constant, the circuital relation of Ampère, now of type

$$4\pi u = d\gamma/dy - d\beta/dz,$$

necessitates, as regards free aether, and in fact at all places where the total current is one of aethereal displacement, the relation

$$(\alpha, \beta, \gamma)/4\pi = (qh - rg, rf - ph, pg - qf) \\ - (d/dx, d/dy, d/dz)\phi,$$

in which ϕ is an undetermined function continuous as to itself and its gradient except at the surfaces of transition; this is clearly the most general value of (α, β, γ) which is consistent with that circuital relation. The part of it depending on ϕ includes the extraneous magnetic field, and also the field due to magnets, if any, that belong to the material system itself, as well as the magnetic field due to the convection of the electric charges on the conductors. It will appear in §§ 99, 100 that this convective effect may safely be neglected, so that $-4\pi\phi$ is simply the magnetic potential of all the magnets in the field. Combining the relation thus obtained between (f, g, h) and (a, b, c) with the direct constitutive relation of type

$$4\pi C^2 f = P - qc + rb,$$

we have

$$C^2 f = P/4\pi - q(pg - qf) + r(rf - ph) + q\,d\phi/dz - r\,d\phi/dy$$

that is

$$(C^2 - p^2 - q^2 - r^2)f = P/4\pi - p(pf + qg + rh) + q\,d\phi/dz - r\,d\phi/dy \\ = P/4\pi - p/4\pi C^2(pP + qQ + rR) + q\,d\phi/dz - r\,d\phi/dy,$$

wherein as above

$$(P, Q, R) = -(d/dx, d/dy, d/dz)V.$$

96. Now the total current is always a stream, just as much flowing out of any region as flows into it; therefore in free space

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0,$$

which in fact merely expresses the electric incompressibility of the aether. Hence finally we obtain for free space

$$\nabla^2 V=C^{-2}\left(p\frac{d}{dx}+q\frac{d}{dy}+r\frac{d}{dz}\right)^2 V,$$

in which ϕ has disappeared, as the characteristic equation from which the single independent variable V of the problem is to be determined, subject to the condition that it is to be constant over each conductor (§ 94) inasmuch as the surface-charge is in equilibrium. When V has thus been determined, its gradient is the electric force (P, Q, R); and the value of (f, g, h) is then given by the equations above in terms of the latter and ϕ, and finally (α, β, γ) or (a, b, c) is obtained in terms of (f, g, h) and ϕ.

For the interior of the conductors V is constant and the electric force (P, Q, R) vanishes: yet the aethereal displacement (f, g, h) *does not vanish* in the conductors, being now given by equations of type

$$(C^2-p^2-q^2-r^2)f=q\,d\phi/dz-r\,d\phi/dy,$$

which make it circuital, so that there is no volume-density of electrification.

In an investigation in detail of the change in the free electric distribution produced by the motion, it will conduce to brevity to take (p, q, r) to be a velocity v parallel to the axis of x. The characteristic equation for V is then

$$\nabla^2 V=\frac{v^2}{C^2}\frac{d^2V}{dx^2},$$

subject to V being constant over each conductor; as the change in the form of this equation arising from the motion depends on $(v/C)^2$, the differences thereby introduced will all be of the

second order of small quantities. Thus, ϵ denoting $(1 - v^2/C^2)^{-1}$, we have

$$\frac{d^2V}{dx'^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0,$$

provided $$dx' = \epsilon^{\frac{1}{2}} dx.$$

Now imagine a correlative material system such that to the point (x, y, z) of the actual system corresponds the point (x', y, z) or $(\epsilon^{\frac{1}{2}} x, y, z)$ of the new one, and solve the problem of free electric distribution on the conductors of the new system supposed at rest; then the electric potentials of the actual moving system and of this stationary system will be the same at corresponding points in the surrounding free aether. The charges in corresponding elements of volume will be proportional: for these may be considered as expressed by a volume density equal to the concentration of the aethereal displacement (f, g, h), there being no electric polarization, while we obtain from the expression for this displacement

$$4\pi C^2 \left(1 - \frac{v^2}{C^2}\right)\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right) = \left(1 - \frac{v^2}{C^2}\right) \frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz}$$

$$= -\left(\frac{d^2V}{dx'^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}\right).$$

The electric charge in any region of the moving system is thus $\epsilon^{1/2}$ times that in the corresponding region of the correlative system at rest. To institute a correspondence for equal charges in the two systems we must multiply the field of the stationary system by $\epsilon^{1/2}$. The magnetic force at any point where the current is wholly one of aethereal displacement is as above $4\pi(0, -vh, vg)$ together with the gradient of a magnetic potential $-4\pi\phi$.

97. A simple example is afforded by the case of a single isolated spherical conductor: the field arising from a free charge Q on the sphere

$$x^2 + y^2 + z^2 = r^2$$

moving with velocity v along the axis of x will correspond to that of a charge $\epsilon^{-\frac{1}{2}}Q$ on the ellipsoid

$$\epsilon^{-1}x^2 + y^2 + z^2 = r^2$$

at rest. The charge on the moving sphere will thus be uniformly distributed over its surface*. The lines of electric force in the surrounding space, which are to be considered as carried on steadily by the motion of the sphere, will not be radial in the immediate neighbourhood, but their level surfaces will ~~will be the curves~~ linearly correspond~~ing~~ to those of ~~the lines of force of~~ the steady distribution on this stationary ellipsoid. At a distance large compared with the dimensions of the stationary ellipsoid its lines of force will however be sensibly radial, and uniformly distributed around it: hence at a distance from the moving sphere its lines of electric force will be ~~radial but~~ concentrated towards that diametral plane which is at right angles to the direction of motion.

In the absence of an extraneous magnetic field, the aethereal displacement around the moving system (on which the statical energy directly depends) is connected with the electric force by the relation

$$4\pi C^2 (f, g, h) = (P, \ \epsilon Q, \ \epsilon R);$$

thus, ϵ being greater than unity, the displacement is still more concentrated towards the diametral plane than is the electric force.

In the case of a conductor of any form with charge e, in uniform motion of translation, the electric force of its steady field, at places in the surrounding free aether whose distance is great compared with the linear dimensions of the conductor, is

$$-\epsilon^{-\frac{1}{2}}\left(\epsilon^{-1}\frac{d}{dx}, \ \frac{d}{dy}, \ \frac{d}{dz}\right)\frac{e}{r},$$

while the aethereal displacement is

$$-\epsilon^{\frac{1}{2}}(4\pi C^2)^{-1}\left(\epsilon^{-1}\frac{d}{dx}, \ \frac{d}{dy}, \ \frac{d}{dz}\right)\frac{e}{r}.$$

* It is easy to see that the distribution of a charge on an ellipsoid moving with any uniform velocity of translation will also be the same as if it were at rest.

98. The main general result is that, whatever be the extraneous or imposed magnetic field, the distribution of charges on the moving system of conductors is identical with that of equal charges on a stationary system which is the same as the actual one uniformly elongated in the ratio $\epsilon^{\frac{1}{2}}$ or $1+\frac{1}{2}v^2/C^2$ in the direction of motion. Now on a physical hypothesis to be presently discussed (§ 112) one effect of the motion is to actually cause a material system to shrink in this direction in the ratio $\epsilon^{-\frac{1}{2}}$. Combining these statements, and neglecting $(v/C)^4$, the actual shrinkage cancels this hypothetical elongation, and we reach the conclusion that when the material system is put into steady uniform motion it shrinks in this ratio $\epsilon^{-\frac{1}{2}}$ in the direction of the motion, while the electric distribution throughout it and the distribution of electric ~~force~~ potential around it remain the same as if it were at rest. This constitutes a direct verification for the special case here under consideration, of general results to be developed subsequently (§ 112).

99. We have still to trace the change of the magnetic field, which will involve the determination of ϕ. Throughout the field

$$\frac{a}{4\pi} = qh - rg - \frac{d\phi}{dx}$$

$$= \frac{qR - rQ}{4\pi\epsilon^2} - \left(1 + \frac{p^2+q^2+r^2}{C^2}\right)\frac{d\phi}{dx} + \frac{p}{C^2}\left(p\frac{d\phi}{dx} + q\frac{d\phi}{dy} + r\frac{d\phi}{dz}\right),$$

with similar expressions for β and γ, correct up to the second order, where the force (P, Q, R) is the gradient of an electric potential. The characteristic equation for ϕ is now to be derived from the stream condition

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0;$$

it does not involve V: it is to be solved so as to preserve the suitable continuity across the surface, namely that of tangential aethereal displacement.

Thus taking for simplicity the velocity v of the system to be parallel to the axis of x, we have

$$\frac{\alpha}{4\pi} = -\frac{d\phi}{dx},$$

$$\frac{\beta}{4\pi} = \frac{v}{4\pi C^2}\frac{dV}{dz} - \left(1 + \frac{v^2}{C^2}\right)\frac{d\phi}{dy},$$

$$\frac{\gamma}{4\pi} = -\frac{v}{4\pi C^2}\frac{dV}{dy} - \left(1 + \frac{v^2}{C^2}\right)\frac{d\phi}{dz};$$

so that

$$\frac{d^2\phi}{dx'^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0,$$

where $$x' = \epsilon^{\frac{1}{2}} x.$$

Hence ϕ is a Laplacian function of (x', y, z) inside each conductor, and another such function of (x', y, z) in the surrounding free space, having poles where the magnetism is situated, these functions being determined by satisfying the condition of continuity of tangential aethereal displacement, that is, of aethereal stress, at the moving interfaces.

100. Let us consider first the case when there are no permanent magnets in the field: if ϕ were then devoid of discontinuity at each interface it must be identically null. Now the effect of assuming such continuity would be to derange the distribution of aethereal displacement only in the second order. Thus up to the first order the assumption is justified: moreover, if the form of the material system is considered as altered by its motion through the aether in the manner of § 112 *infra*, it may be verified that there is no discrepancy even of the second order. Hence the motion of the system produces no effect on the electric force or the electric distribution in it; while the magnetic field is $v/C^2 \,.\, (0, -R, Q)$ and the aethereal displacement is augmented by the second-order term $(4\pi)^{-1}(v/C^2)^2(0, Q, R)$, in agreement with the general theory of Chapter X.

If there are permanent magnets convected along with the steadily moving electric system, this argument still proves that any electric effect depending on ϕ is transmitted wholly from them, and does not involve a part arising from convection of the electric charges. But the considerations of §41 shew that the electric force arising from their motion is masked by an induced electrification in the magnets themselves. Thus, even up to the second order, the convection, along with the Earth in its orbital motion, of a powerful magnet, either itself conducting or surrounded by a conducting screen, will not produce any effect on electric distributions in neighbouring bodies by introducing a new term into the electric force, as has sometimes been suggested.

As, on the above hypothesis of a very minute material deformation of a moving system when it is put into uniform motion of translation, the electric ~~force~~ distribution remains absolutely unaltered at each point in its neighbourhood, it follows that those mutual mechanical forces between the electrically charged conductors forming the system, which arise from the operation of this electric force, are unaltered. We have seen however that the convection produces a magnetic force of the first order depending on the electric force at the place, which we might seek means of detecting in the case when it arises from the motion of the Earth. That cannot be done by deflexion of a magnetic needle, for the needle would experience a counteracting electric distribution over its surface which would annul the electric force inside it, so that the magnetic force of convective origin, acting on its elements, would be annulled also: while we have just recognised that the forces exerted by the surrounding electric charges on the distribution induced on the needle are not affected by the Earth's motion. The result of an experiment depending on the deflexion of a magnetic needle should therefore be negative, as Röntgen has found it to be.

101. It is worth while to definitely formulate the scheme of equations which applies to dielectric masses belonging to the moving system. The Faraday circuital relation gives as before

$$(P,\ Q,\ R) = -(d/dx,\ d/dy,\ d/dz)\,V.$$

Also
$$4\pi C^2 f = P, \qquad 4\pi C^2 f' = (K-1)\,P$$
$$4\pi C^2 g = Q + vc, \quad 4\pi C^2 g' = (K-1)\,Q$$
$$4\pi C^2 h = R - vb, \quad 4\pi C^2 h' = (K-1)\,R$$

while by § 64, $a = \alpha,\ b = \beta + 4\pi v h',\ c = \gamma - 4\pi v g'$.

The total current in the dielectric is made up of a displacement current $-v d/dx\,(f, g, h)$ belonging to the stagnant aether and a current of convection of polarization $+ v d/dx\,(f', g', h')$ arising from the moving dielectric matter.

Hence Ampère's circuital relation gives

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = \frac{v}{C^2}(K-2)\frac{dP}{dx}$$

$$\frac{d\alpha}{dz} - \frac{d\gamma}{dx} = \frac{v}{C^2}(K-2)\frac{dQ}{dx} - \frac{v^2}{C^2}\left(\frac{d\gamma}{dx} - 4\pi v\frac{dg'}{dx}\right)$$

$$\frac{d\beta}{dx} - \frac{d\alpha}{dy} = \frac{v}{C^2}(K-2)\frac{dR}{dx} + \frac{v^2}{C^2}\left(\frac{d\beta}{dx} + 4\pi v\frac{dh'}{dx}\right).$$

On elimination of (α, β, γ) we obtain, after some reductions, as the characteristic equation of the electric potential,

$$\left(1 - \frac{v^2}{C^2}\right)\frac{d^2V}{dx^2} + \left(1 + \frac{K-1}{K \pm 2}\frac{v^2}{C^2}\right)\left(\frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}\right) = 0.$$

It is however futile in this mode of procedure to attempt to carry the approximation beyond the first order of v/C, as the value (and form) of K will be itself affected to the second order in a manner as yet unknown. Up to this order V will satisfy the same characteristic equation as if the system were at rest; as it is constant over each conductor, it follows that the electric force will be the same everywhere as if the system were at rest. Also up to this order the magnetic field (α, β, γ) will be that arising from the imposed magnetic system together with a distribution of currents derived from a flow-potential

$$-\frac{v}{4\pi C^2}(K-2)\frac{dV}{dx}.$$

Case of uniform rotation.

101**. In further illustration of the general principles, the case of rotation round a fixed axis in a symmetrical magnetic field will be briefly considered. The axes of coordinates will now be taken fixed in the aether. In any steady state the electric force (P, Q, R), given by $P = qc - rb - dF/dt - d\Psi/dx$, is derived from a potential V. Thus

$$d(\Psi - V) = \begin{vmatrix} dx & dy & dz \\ p & q & r \\ a & b & c \end{vmatrix} = \begin{vmatrix} dr & rd\theta & dz \\ 0 & \omega r & 0 \\ a & b & c \end{vmatrix},$$

the second determinant corresponding to the present problem with columnar coordinates. When the first determinant is not an exact differential, the steady motion must involve electric flow. In the present case

$$\Psi - V = \omega \int (crdr - ardz);$$

so that (a, c) should be derived from a stream function, as it is by the circuital quality of (a, b, c) whenever b is the same all round the axis. Further in this case

$$\nabla^2 (\Psi - V) = \omega r \left(\frac{dc}{dr} - \frac{da}{dz} + \frac{2c}{r}\right),$$

which is equal to $2\omega c$ at places where there is no current.

In a conductor V is constant because electric force would induce compensating charge: thus (§ 70) the electrification in it is that belonging to the static potential Ψ here determined, involving a volume-density $-\omega c/2\pi C^2$ as well as a surface distribution. Outside the conductor the circuitality of the aethereal displacement requires that $\nabla^2\Psi$ is null: and Ψ must be itself continuous across the surface because there cannot be discontinuity in the aether-strain. Moreover, in Ψ is included the potential of the steady influencing electric field arising from neighbouring stationary electrified bodies: but it is V, or $\Psi - \omega \int (crdr - ardz)$, and not Ψ, that is constant over the surface of the conductor.

In a dielectric portion of the rotating system, (f'', g'', h'') is

circuital, while $4\pi C^2 f' = -(K-1)\,dV/dx$ and $4\pi C^2 f = -d\Psi/dx$; thus $\nabla^2\Psi$ is equal to $(1-K^{-1})\nabla^2(\Psi - V)$ and is therefore known, being null in free aether. If there is no surface charge on the dielectric body, $K\dfrac{d\Psi}{dn} - (K-1)\dfrac{d(\Psi - V)}{dn}$ must be continuous across the surface, as well as Ψ itself: thus the effect on the surrounding electric field of the rotation of the dielectric body is the same as would be that of an electric charge in it of volume-density $-(K-1)\,\omega c/2\pi C^2$, and surface-density $\dfrac{K-1}{4\pi C^2}\dfrac{d(\Psi - V)}{dn}$ of which the last factor is the tangential magnetic force just outside multiplied by ωr. The magnetic effect of the rotating polarization of the dielectric body may be directly calculated, up to the first order, as that due to its equivalent current-system, of intensity $(K-1)\dfrac{\omega}{4\pi C^2}\dfrac{dV}{dr}$ flowing in circles around the axis.

It may be verified that in a charged solid spherical conductor of radius a, rotating in a uniform magnetic field c, and referred to polar coordinates (ρ, ϕ) measured from the axis of rotation which is that of the field, $\Psi = C + \frac{1}{2}\omega c\rho^2 \sin^2\phi$, thus involving an electric volume-density $-\omega c/2\pi C^2$ and surface-density $\dfrac{C}{4\pi a C^2} + \dfrac{\omega c a}{8\pi C^2}(5\sin^2\phi - \frac{4}{3})$, in which C is determined by the amount of the charge when the sphere is insulated, or by the point of it which is in connexion with Earth, being null when the axis of rotation is uninsulated*. This approximation may be improved, if it is so desired, by including the inappreciable modification of the extraneous magnetic field arising from the convection of the electric charge of the sphere.

The general case of steady spiral motion, including that of uniform translation, may also be treated in this manner.

* *Phil. Mag.*, Jan. 1884, p. 4: cf. also *Phil. Trans.* 1895 A, pp. 727—31.

CHAPTER X

GENERAL PROBLEM OF MOVING MATTER TREATED IN RELATION TO THE INDIVIDUAL MOLECULES

Formulation of the Problem

102. We shall now consider the material system as consisting of free aether pervaded by a system of electrons which are to be treated individually, some of them free or isolated, but the great majority of them grouped into material molecules: and we shall attempt to compare the relative motions of these electrons when they form, or belong to, a material system devoid of translatory motion through the aether, with what it would be when a translatory velocity is superposed, say for shortness a velocity v parallel to the axis of x. The medium in which the activity occurs is for our present purpose the free aether itself, whose dynamical equations have been definitely ascertained in quite independent ways from consideration of both the optical side and the electrodynamic side of its activity: so that there will be nothing hypothetical in our analysis on that score. An electron e will occur in this analysis as a singular point in the aether, on approaching which the elastic strain constituting the aethereal displacement (f, g, h) increases indefinitely, according to the type

$$-e/4\pi \,.\, (d/dx,\ d/dy,\ d/dz)\, r^{-1}:$$

it is in fact analogous to what is called a simple pole in the two-dimensional representation that is employed in the theory of a function of a complex variable. It is assumed that this singularity represents a definite structure, forming a nucleus of strain in the aether, which is capable of transference across that medium independently of motion of the aether itself: the

portion of the surrounding aethereal strain, of which the displacement-vector (f, g, h) is the expression, which is associated with the electron and is carried along with the electron in its motion, being as above $-e/4\pi \,.\, (d/dx, d/dy, d/dz)r^{-1}$. It is to be noticed that the energy of this part of the displacement is closely concentrated around the nucleus of the electron, and not widely diffused as might at first sight appear. The aethereal displacement satisfies the stream-condition

$$df/dx + dg/dy + dh/dz = 0,$$

except where there are electrons in the effective element of volume: these are analogous to the so-called sources and sinks in the abstract theory of liquid flow, so that when electrons are present the integral of the normal component of the aethereal displacement over the boundary of any region, instead of being null, is equal to the quantity Σe of electrons existing in the region. The other vector which is associated with the aether, namely the magnetic induction (a, b, c), also possesses the stream property; but singular points in its distribution, of the nature of simple poles, do not exist. The motion of an electron involves however a singularity in (a, b, c), of a rotational type, with its nucleus at the moving electron**; and the time-average of this singularity for a very rapid minute steady orbital motion of an electron is analytically equivalent, at distances considerable compared with the dimensions of the orbit, to a magnetic doublet analogous to a source and

** Namely as the distance r from it diminishes indefinitely, the magnetic induction tends to the form $evr^{-2} \sin \theta$, at right angles to the plane of the angle θ between r and the velocity v of the electron: this arises as the disturbance of the medium involved in annulling the electron in its original position and restoring it in the new position to which it has moved. The relations will appear more clearly when visualized by the kinematic representation of Appendix E; or when we pass to the limit in the formulae of Chapter ix relating to the field of a moving charged body of finite dimensions.

The specification in the text, as a simple pole, only applies for an electron moving with velocity v, when terms of the order $(v/c)^2$ are neglected: otherwise the aethereal field close around it is not isotropic and an amended specification derivable from the formulae of Chapter ix must be substituted. In the second-order discussion of Chapter xi this more exact form is implicitly involved, the strength of the electron being determined (§ 111) by the concentration of the aethereal displacement around it. The singularity in the magnetic field which is involved in the motion of the electron, not of course an intrinsic one, has no concentration.

associated equal sink. Finally, the various parts of the aether are supposed to be sensibly at rest, so that for example the time-rate of change of the strain of any element of the aether is represented by differentiation with respect to the time without any additional terms to represent the change due to the element of aether being carried on in the meantime to a new position; in this respect the equations of the aether are much simpler than those of the dynamics of fluid motion, being in fact linear. The aether is stagnant on this theory, while the molecules constituting the Earth and all other material bodies flit through it without producing any finite flow in it; hence the law of the astronomical aberration of light is rigorously maintained, and the Doppler change of wave-length of radiation from a moving source holds good; but it will appear that all purely terrestrial optical phenomena are unaffected by the Earth's motion.

103. Subject to this general explanation, the analytical equations which express the dynamics of the field of free aether, existing between and around the nuclei of the electrons, are

$$4\pi \frac{d}{dt}(f, g, h) = \operatorname{curl}(a, b, c)$$

$$-\frac{d}{dt}(a, b, c) = 4\pi C^2 \operatorname{curl}(f, g, h),$$

in which the symbol curl (a, b, c) represents, after Maxwell, the vector

$$\left(\frac{dc}{dy} - \frac{db}{dz},\ \frac{da}{dz} - \frac{dc}{dx},\ \frac{db}{dx} - \frac{da}{dy}\right),$$

and in which C is the single physical constant of the aether, being the velocity of propagation of elastic disturbances through it. These are the analytical equations derived by Maxwell in his mathematical development of Faraday's views as to an electric medium: and they are the same as the equations arrived at by MacCullagh a quarter of a century earlier in his formulation of the dynamics of optical media. It may fairly be claimed that the theoretical investigations of Maxwell, in combination with the experimental verifications of Hertz and his successors in that field, have imparted to this analytical formulation of the dynamical relations of free aether an exact-

ness and precision which is not surpassed in any other department of physics, even in the theory of gravitation.

Where a more speculative element enters is in the construction of a kinematic scheme of representation of the aether-strain, such as will allow of the unification of the various assumptions here enumerated. It is desirable for the sake of further insight, and even necessary for various applications, to have concrete notions of the physical nature of the vectors (f, g, h) and (a, b, c) which specify aethereal disturbances, in the form of representations such as will implicitly and intuitively involve the analytical relations between them, and will also involve the conditions and restrictions to which each is subject, including therein the permanence and characteristic properties of an electron and its free mobility through the aether*.

104. But for the mere analytical development of the aether-scheme as above formulated, a concrete physical representation of the constitution of the aether is not required: the abstract relations and conditions above given form a sufficient basis. In point of fact these analytical relations are theoretically of an ideal simplicity for this purpose: for they give explicitly the time-rates of change of the vectors of the problem at each instant, so that from a knowledge of the state of the system at any time t the state at the time $t + \delta t$ can be immediately expressed, and so by successive steps, or by the use of Taylor's differential expansion-theorem, its state at any further time can theoretically be derived. The point that requires careful attention is as to whether the solution of these equations in terms of a given initial state of the system determines the motions of the electrons or strain-nuclei through the medium, as well as the changes of strain in the medium itself: and it will appear on consideration that under suitable hypotheses this is so. For the given initial state will involve given motions of the electrons, that is the initial value of (a, b, c) will involve rotational singularities at the electrons around their directions of motion, just such as in the element of time δt will shift the electrons themselves into their new positions**: and so on step by step continually. This however

* See Appendix E. ** Cf. footnote, p. 162.

presupposes that the nucleus of the electron is quite labile as regards displacement through the aether, in other words that its movement is not influenced by any inertia or forces except such as are the expression of its relation to the aether: we in fact assume the *completeness* of the aethereal scheme of relations as above given. Any difficulty that may be felt on account of the infinite values of the vectors at the nucleus itself may be removed, in the manner customary in analytical discussions on attractions, by considering the nucleus to consist of a volume distribution of electricity of finite but very great density, distributed through a very small space instead of being absolutely concentrated in a point: then the quantities will not become infinite. Of the detailed structure of electrons nothing is assumed: so long as the actual dimensions of their nuclei are extremely small in comparison with the distances between them, it will suffice for the theory to consider them as points, just as for example in the general gravitational theory of the Solar System it suffices to consider the planets as attracting points. This method is incomplete only as regards those portions of the energy and other quantities that are associated with the mutual actions of the parts of the electron itself, and are thus molecularly constitutive.

105. It is to be observed that on the view here being developed, in which atoms of matter are constituted of aggregations of electrons, the only actions between atoms are what may be described as electric forces. The electric character of the forces of chemical affinity was an accepted part of the chemical views of Davy, Berzelius, and Faraday; and more recent discussions, while clearing away crude conceptions, have invariably tended to the strengthening of that hypothesis. The mode in which the ordinary forces of cohesion could be included in such a view is still quite undeveloped. Difficulties of this kind have however not been felt to be fundamental in the vortex-atom illustration of the constitution of matter, which has exercised much fascination over high authorities on molecular physics: yet in the concrete realization of Maxwell's theory of the aether above referred to, the atom of matter

possesses all the dynamical properties of a vortex ring in a frictionless fluid, so that everything that can be done in the domain of vortex-ring illustration is implicitly attached to the present scheme. The fact that virtually nothing has been achieved in the department of forces of cohesion is not a valid objection to the development of a theory of the present kind. For the aim of theoretical physics is not a complete and summary conquest of the *modus operandi* of natural phenomena: that would be hopelessly unattainable if only for the reason that the mental apparatus with which we conduct the search is itself in one of its aspects a part of the scheme of Nature which it attempts to unravel. But the very fact that this is so is evidence of a correlation between the process of thought and the processes of external phenomena, and is an incitement to push on further and bring out into still clearer and more direct view their inter-connexions. When we have mentally reduced to their simple elements the correlations of a large domain of physical phenomena, an objection does not lie because we do not know the way to push the same principles to the explanation of other phenomena to which they should presumably apply, but which are mainly beyond the reach of our direct examination.

The natural conclusion would rather be that a scheme, which has been successful in the simple and large-scale physical phenomena that we can explore in detail, must also have its place, with proper modifications or additions on account of the difference of scale, in the more minute features of the material world as to which direct knowledge in detail is not available. And in any case, whatever view may be held as to the necessity of the whole complex of chemical reaction being explicable in detail by an efficient physical scheme, a limit is imposed when vital activity is approached: any complete analysis of the conditions of the latter, when merely superficial sequences of phenomena are excluded, must remain outside the limits of our reasoning faculties. The object of scientific explanation is in fact to coordinate mentally, but not to exhaust, the interlaced maze of natural phenomena: a theory which gives an adequate correlation of a portion of this field maintains its

place until it is proved to be in definite contradiction, not removable by suitable modification, with another portion of it.

Application to moving Material Media: approximation up to first order

106. We now recall the equations of the free aether, with a view to changing from axes (x, y, z) at rest in the aether to axes (x', y', z') moving with translatory velocity v parallel to the axis of x; so as thereby to be in a position to examine how phenomena are altered when the observer and his apparatus are in uniform motion through the stationary aether. These equations are

$$4\pi\frac{df}{dt}=\frac{dc}{dy}-\frac{db}{dz} \qquad -(4\pi C^2)^{-1}\frac{da}{dt}=\frac{dh}{dy}-\frac{dg}{dz}$$

$$4\pi\frac{dg}{dt}=\frac{da}{dz}-\frac{dc}{dx} \qquad -(4\pi C^2)^{-1}\frac{db}{dt}=\frac{df}{dz}-\frac{dh}{dx}$$

$$4\pi\frac{dh}{dt}=\frac{db}{dx}-\frac{da}{dy} \qquad -(4\pi C^2)^{-1}\frac{dc}{dt}=\frac{dg}{dx}-\frac{df}{dy}.$$

When they are referred to the axes (x', y', z') in uniform motion, so that $(x', y', z')=(x-vt,\ y,\ z)$, $t'=t$, then d/dx, d/dy, d/dz become d/dx', d/dy', d/dz', but d/dt becomes $d/dt'-vd/dx'$: thus

$$4\pi\frac{df}{dt'}=\frac{dc'}{dy'}-\frac{db'}{dz'} \qquad -(4\pi C^2)^{-1}\frac{da}{dt'}=\frac{dh'}{dy'}-\frac{dg'}{dz'}.$$

$$4\pi\frac{dg}{dt'}=\frac{da'}{dz'}-\frac{dc'}{dx'} \qquad -(4\pi C^2)^{-1}\frac{db}{dt'}=\frac{df'}{dz'}-\frac{dh'}{dx'}$$

$$4\pi\frac{dh}{dt'}=\frac{db'}{dx'}-\frac{da'}{dy'} \qquad -(4\pi C^2)^{-1}\frac{dc}{dt'}=\frac{dg'}{dx'}-\frac{df'}{dy'},$$

where

$$(a',\ b',\ c')=(a,\ b+4\pi vh,\ c-4\pi vg)$$

$$(f',\ g',\ h')=\left(f,\ g-\frac{v}{4\pi C^2}c,\ h+\frac{v}{4\pi C^2}b\right).$$

We can complete the elimination of (f, g, h) and (a, b, c) so

that only the vectors denoted by accented symbols shall remain, by substituting from these latter formulae: thus

$$g = g' + \frac{v}{4\pi C^2}(c' + 4\pi v g),$$

so that
$$\epsilon^{-1} g = g' + \frac{v}{4\pi C^2} c',$$

where ϵ is equal to $(1 - v^2/C^2)^{-1}$, and exceeds unity;

and
$$b = b' - 4\pi v \left(h' - \frac{v}{4\pi C^2} b\right)$$

so that
$$\epsilon^{-1} b = b' - 4\pi v h';$$

giving the general relations

$$\epsilon^{-1}(a,\ b,\ c) = (\epsilon^{-1} a',\ b' - 4\pi v h',\ c' + 4\pi v g')$$

$$\epsilon^{-1}(f, g, h) = \left(\epsilon^{-1} f',\ g' + \frac{v}{4\pi C^2} c',\ h' - \frac{v}{4\pi C^2} b'\right).$$

Hence

$$4\pi \frac{df'}{dt'} = \frac{dc'}{dy'} - \frac{db'}{dz'}$$

$$4\pi\epsilon \frac{dg'}{dt'} = \frac{da'}{dz'} - \left(\frac{d}{dx'} + \frac{v}{C^2}\epsilon\frac{d}{dt'}\right) c'$$

$$4\pi\epsilon \frac{dh'}{dt'} = \left(\frac{d}{dx'} + \frac{v}{C^2}\epsilon\frac{d}{dt'}\right) b' - \frac{da'}{dy'}$$

$$-(4\pi C^2)^{-1} \frac{da'}{dt'} = \frac{dh'}{dy'} - \frac{dg'}{dz'}$$

$$-(4\pi C^2)^{-1}\epsilon \frac{db'}{dt'} = \frac{df'}{dz'} - \left(\frac{d}{dx'} + \frac{v}{C^2}\epsilon\frac{d}{dt'}\right) h'$$

$$-(4\pi C^2)^{-1}\epsilon \frac{dc'}{dt'} = \left(\frac{d}{dx'} + \frac{v}{C^2}\epsilon\frac{d}{dt'}\right) g' - \frac{df'}{dy'}.$$

Now change the time-variable from t' to t'', equal to $t' - \frac{v}{C^2}\epsilon x'$; this will involve that $\frac{d}{dx'} + \frac{v}{C^2}\epsilon\frac{d}{dt'}$ is replaced by $\frac{d}{dx'}$, while the other differential operators remain unmodified; thus the scheme of equations reverts to the same type as when it was referred to axes at rest, except as regards the factors ϵ on the left-hand sides.

107. It is to be observed that this factor ϵ only differs from unity by $(v/C)^2$, which is of the second order of small quantities; hence we have the following correspondence when that order is neglected. Consider any aethereal system, and let the sequence of its spontaneous changes referred to axes (x', y', z') moving uniformly through the aether with velocity $(v, 0, 0)$ be represented by values of the vectors (f, g, h) and (a, b, c) expressed as functions of x', y', z' and t', the latter being the time measured in the ordinary manner: then there exists a correlated aethereal system whose sequence of spontaneous changes referred to axes (x', y', z') at rest are such that its electric and magnetic vectors (f', g', h') and (a', b', c') are functions of the variables x', y', z' and a time-variable t'', equal to $t' - \frac{v}{C^2}x'$, which are the same as represent the quantities

$$\left(f,\ g - \frac{v}{4\pi C^2}c,\ h + \frac{v}{4\pi C^2}b\right)$$

and $$(a,\ b + 4\pi vh,\ c - 4\pi vg)$$

belonging to the related moving system when expressed as functions of the variables x', y', z' and t'.

Conversely, taking any aethereal system at rest in the aether, let the sequence of its changes be represented by (f', g', h') and (a', b', c') expressed as functions of the coordinates (x, y, z) and of the time t'. In these functions change t' into $t - \frac{v}{C^2}x$: then the resulting expressions are the values of

$$\left(f,\ g - \frac{v}{4\pi C^2}c,\ h + \frac{v}{4\pi C^2}b\right),$$

and $$(a,\ b + 4\pi vh,\ c - 4\pi vg),$$

for a system in uniform motion through the aether, referred to axes (x, y, z) moving along with it, and to the time t. In comparing the states of the two systems, we have to the first order

$$\frac{df}{dx} \quad \text{equal to} \quad \frac{df'}{dx} - \frac{v}{C^2}\frac{df'}{dt}$$

$$\frac{d}{dy}\left(g - \frac{v}{4\pi C^2}c\right) \quad ,, \quad \frac{dg'}{dy}$$

$$\frac{d}{dz}\left(h + \frac{v}{4\pi C^2}b\right) \quad ,, \quad \frac{dh'}{dz};$$

hence bearing in mind that for the system at rest

$$\frac{dc'}{dy} - \frac{db'}{dz} = 4\pi \frac{df'}{dt'},$$

or, what is the same,

$$\frac{dc'}{dy} - \frac{db'}{dz} = 4\pi \left(\frac{df'}{dt} - v\frac{df'}{dx}\right),$$

we have, to the first order,

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = \frac{df'}{dx} + \frac{dg'}{dy} + \frac{dh'}{dz}.$$

Thus the electrons in the two systems here compared, being situated at the singular points at which the concentration of the electric displacement ceases to vanish, occupy corresponding positions. Again, these electrons are of equal strengths: for, very near an electron, fixed or moving, the values of (f, g, h) and (a, b, c) are practically those due to it, the part due to the remainder of the system being negligible in comparison: also in this correspondence the relation between (f, g, h) and the accented variables is, by § 106

$$\epsilon^{-1}(f, g, h) = \left(\epsilon^{-1}f',\ g' + \frac{v}{4\pi C^2}c',\ h' - \frac{v}{4\pi C^2}b'\right);$$

hence, since for the single electron at rest (a', b', c') is null, we have, very close to the correlative electron in the moving system, (f, g, h) equal to $(f', \epsilon g', \epsilon h')$, where ϵ, being $(1 - v^2/C^2)^{-1}$, differs from unity by the second order of small quantities. Thus neglecting the second order, (f, g, h) is equal to (f', g', h') for corresponding points very close to electrons; and, as the amount of electricity inside any boundary is equal to the integral of the normal component of the aethereal displacement taken over the boundary, it follows by taking a very contracted boundary that the strengths of the corresponding electrons in the two systems are the same, to this order of approximation.

108. It is to be observed that the above analytical transformation of the equations applies to any isotropic dielectric medium as well as to free aether: we have only to alter C into

the velocity of radiation in that medium, and all will be as above. The transformation will thus be different for different media. But we are arrested if we attempt to proceed to compare a moving material system, treated as continuous, with the same system at rest; for the motion of the polarized dielectric matter has altered the mathematical type of the electric current. It is thus of no avail to try to effect in this way a direct general transformation of equations of a material medium in which dielectric and conductive coefficients occur.

109. The correspondence here established between a system referred to fixed axes and a system referred to moving axes will assume a very simple aspect when the former system is a steady one, so that the variables are independent of the time. Then the distribution of electrons in the second system will be at each instant precisely the same as that in the first, while the second system accompanies its axes of reference in their uniform motion through the aether. In other words, given any system of electrified bodies at rest, in equilibrium under their mutual electric influences and imposed constraints, there will be a precisely identical system in equilibrium under the same constraints, and in uniform translatory motion through the aether. That is, uniform translatory motion through the aether does not produce any alteration in electric distributions as far as the first order of the ratio of the velocity of the system to the velocity of radiation is concerned. Various cases of this general proposition will be verified subsequently in connexion with special investigations.

Moreover this result is independent of any theory as to the nature of the forces between material molecules: the structure of the matter being assumed unaltered to the first order by motion through the aether, so too must be all electric distributions. What has been proved comes to this, that if any configuration of ionic charges is the natural one in a material system at rest, the maintenance of the same configuration as regards the system in uniform motion will not require the aid of any new forces. The electron *taken by itself* must be on any conceivable theory a simple singularity of the aether whose

movements when it is free, and interactions with other electrons if it can be constrained by matter, are traceable through the differential equations of the surrounding free aether alone: and a correlation has been established between these equations for the two cases above compared. It is however to be observed (cf. § 99) that though the fixed and the moving system of electrons of this correlation are at corresponding instants identical, yet the electric and magnetic displacements belonging to them differ by terms of the first order.

CHAPTER XI

MOVING MATERIAL SYSTEM: APPROXIMATION CARRIED TO THE SECOND ORDER

110. THE results above obtained have been derived from the correlation developed in § 106, up to the first order of the small quantity v/C, between the equations for aethereal vectors here represented by (f', g', h') and (a', b', c') referred to the axes (x', y', z') at rest in the aether and a time t'', and those for related aethereal vectors represented by (f, g, h) and (a, b, c) referred to axes (x', y', z') in uniform translatory motion and a time t'. But we can proceed further, and by aid of a more complete transformation institute a correspondence which will be correct to the second order. Writing as before t'' for $t' - \frac{v}{C^2}\epsilon x'$, the exact equations for (f, g, h) and (a, b, c) referred to the moving axes (x', y', z') and time t' are, as above shown, equivalent to

$$4\pi \frac{df'}{dt''} = \frac{dc'}{dy'} - \frac{db'}{dz'} \qquad -(4\pi C^2)^{-1} \frac{da'}{dt''} = \frac{dh'}{dy'} - \frac{dg'}{dz'}$$

$$4\pi\epsilon \frac{dg'}{dt''} = \frac{da'}{dz'} - \frac{dc'}{dx'} \qquad -(4\pi C^2)^{-1}\epsilon \frac{db'}{dt''} = \frac{df'}{dz'} - \frac{dh'}{dx'}$$

$$4\pi\epsilon \frac{dh'}{dt''} = \frac{db'}{dx'} - \frac{da'}{dy'} \qquad -(4\pi C^2)^{-1}\epsilon \frac{dc'}{dt''} = \frac{dg'}{dx'} - \frac{df'}{dy'}.$$

Now write

(x_1, y_1, z_1) for $(\epsilon^{\frac{1}{2}} x', y', z')$

(a_1, b_1, c_1) for $(\epsilon^{-\frac{1}{2}} a', b', c')$ or $(\epsilon^{-\frac{1}{2}} a, b + 4\pi v h, c - 4\pi v g)$

$$(f_1, g_1, h_1) \text{ for } (\epsilon^{-\frac{1}{2}}f', g', h') \text{ or } \left(\epsilon^{-\frac{1}{2}}f, g - \frac{v}{4\pi C^2}c, h + \frac{v}{4\pi C^2}b\right)$$

$$dt_1 \text{ for } \epsilon^{-\frac{1}{2}}dt'' \qquad \text{or } e^{-\frac{1}{2}}\left(dt' - \frac{v}{C^2}\epsilon dx'\right),$$

where $\epsilon = (1 - v^2/C^2)^{-1}$; and it will be seen that the factor ϵ is absorbed, so that the scheme of equations, referred to moving axes, which connects together the new variables with subscripts, is identical in form with the Maxwellian scheme of relations for the aethereal vectors referred to fixed axes. This transformation, from (x', y', z') to (x_1, y_1, z_1) as dependent variables, signifies an elongation of the space of the problem in the ratio $\epsilon^{\frac{1}{2}}$ along the direction of the motion of the axes of coordinates. Thus if the values of (f_1, g_1, h_1) and (a_1, b_1, c_1) given as functions of x_1, y_1, z_1, t_1 express the course of spontaneous change of the aethereal vectors of a system of moving electrons referred to axes (x_1, y_1, z_1) at rest in the aether, then

$$\left(\epsilon^{-\frac{1}{2}}f, g - \frac{v}{4\pi C^2}c, \quad h + \frac{v}{4\pi C^2}b\right)$$

and
$$(\epsilon^{-\frac{1}{2}}a, \ b + 4\pi vh, \ c - 4\pi vg),$$

expressed by the same functions of the variables

$$\epsilon^{\frac{1}{2}}x', \ y', \ z', \ \epsilon^{-\frac{1}{2}}t' - \frac{v}{C^2}\epsilon^{\frac{1}{2}}x',$$

will represent the course of change of the aethereal vectors (f, g, h) and (a, b, c) of a correlated system of moving electrons referred to axes of (x', y', z') moving through the aether with uniform translatory velocity $(v, 0, 0)$. In this correlation between the courses of change of the two systems, we have

$$\frac{d(\epsilon^{-\frac{1}{2}}f)}{d(\epsilon^{\frac{1}{2}}x')} \text{ equal to } \frac{df_1}{dx_1} - \frac{v}{C^2}\frac{df_1}{dt_1},$$

$$\frac{d}{dy'}\left(g - \frac{v}{4\pi C^2}c\right) \quad \text{,,} \quad \frac{dg_1}{dy_1}$$

$$\frac{d}{dz'}\left(h + \frac{v}{4\pi C^2}b\right) \quad \text{,,} \quad \frac{dh_1}{dz_1},$$

where
$$\frac{dc}{dy'} - \frac{db}{dz'} = 4\pi\left(\frac{df}{dt'} - v\frac{df}{dx'}\right)$$

and also
$$\frac{df_1}{dt_1} = \frac{df}{dt'};$$

hence $\dfrac{df}{dx'} + \dfrac{dg}{dy'} + \dfrac{dh}{dz'} - \dfrac{v}{C^2}\left(\dfrac{df}{dt'} - v\dfrac{df}{dx'}\right)$ is equal to

$$\epsilon\frac{df_1}{dx_1} + \frac{dg_1}{dy_1} + \frac{dh_1}{dz_1} - \frac{v}{C^2}\epsilon\frac{df}{dt'},$$

so that, up to the order of $(v/C)^2$ inclusive,

$$\frac{df}{dx'} + \frac{dg}{dy'} + \frac{dh}{dz'} = \frac{df_1}{dx_1} + \frac{dg_1}{dy_1} + \frac{dh_1}{dz_1}.$$

Thus the conclusions as to the corresponding positions of the electrons of the two systems, which had been previously established up to the first order of v/C, are true up to the second order when the dimensions of the moving system are contracted in comparison with the fixed system in the ratio $\epsilon^{-\frac{1}{2}}$, or $1 - \frac{1}{2}v^2/C^2$, along the direction of its motion.

111. The ratio of the strengths of corresponding electrons in the two systems may now be deduced just as it was previously when the discussion was confined to the first order of v/C. For the case of a single electron in uniform motion the comparison is with a single electron at rest, near which (a_1, b_1, c_1) vanishes so far as it depends on that electron: now we have in the general correlation

$$g = g_1 + \frac{v}{4\pi C^2}(c_1 + 4\pi v g),$$

hence in this particular case

$$(g, h) = \epsilon(g_1, h_1), \text{ while } f = \epsilon^{\frac{1}{2}} f_1.$$

But the strength of the electron in the moving system is the value of the integral $\iint(f dy' dz' + g dz' dx' + h dx' dy')$ extended over any surface closely surrounding its nucleus; that is here $\epsilon^{\frac{1}{2}}\iint(f_1 dy_1 dz_1 + g_1 dz_1 dx_1 + h_1 dx_1 dy_1)$, so that the strength of each moving electron is $\epsilon^{\frac{1}{2}}$ times that of the correlative fixed electron. As before, no matter what other electrons are present, this

argument still applies if the surface be taken to surround the electron under consideration very closely, because then the wholly preponderating part of each vector is that which belongs to the adjacent electron**.

112. We require however to construct a correlative system devoid of the translatory motion in which the strengths of the electrons shall be equal instead of proportional, since motion of a material system containing electrons cannot alter their strengths. The principle of dynamical similarity will effect this.

We have in fact to reduce the scale of the electric ~~charges,~~ volume-densities, and therefore of $\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}$, in a system at rest in the ratio $\epsilon^{-\frac{1}{2}}$. Apply therefore a transformation

$$(x,\ y,\ z)=k(x_1,\ y_1,\ z_1),\quad t=lt_1,$$
$$(a, b, c)=\vartheta(a_1, b_1, c_1),\ (f, g, h)=\epsilon^{-\frac{1}{2}}k(f_1, g_1, h_1);$$

and the form of the fundamental circuital aethereal relations will not be changed provided $k=l$ and $\vartheta=\epsilon^{-\frac{1}{2}}k$. Thus we may have k and l both unity and $\vartheta=\epsilon^{-\frac{1}{2}}$; so that no further change of scale in space and time is required, but only a diminution of (a, b, c) in the ratio $\epsilon^{-\frac{1}{2}}$.

We derive the result, correct to the second order, that if the internal forces of a material system arise wholly from electrodynamic actions between the systems of electrons which constitute the atoms, then an effect of imparting to a steady material system a uniform velocity of translation is to produce a uniform contraction of the system in the direction of the motion, of amount $\epsilon^{-\frac{1}{2}}$ or $\sqrt{1-\frac{1}{2}v^2/C^2}$. The electrons will occupy corresponding positions in this contracted system, but the aethereal displacements in the space around them will not correspond: if (f, g, h) and (a, b, c) are those of the moving system, then the electric and magnetic displacements at corresponding points of the fixed systems will be the values that the vectors

$$\epsilon^{\frac{1}{2}}\left(\epsilon^{-\frac{1}{2}}f,\ g-\frac{v}{4\pi C^2}c,\ h+\frac{v}{4\pi C^2}b\right)$$

** This result follows more immediately from § 110, which shows that corresponding densities of electrification are equal, while corresponding volumes are as $\epsilon^{\frac{1}{2}}$ to unity.

and $$\epsilon^{\frac{1}{2}}(\epsilon^{-\frac{1}{2}}a,\ b+4\pi vh,\ c-4\pi vg)$$

had at a time const. $+ vx/C^2$ before the instant considered when the scale of time is enlarged in the ratio $\epsilon^{\frac{1}{2}}$.

As both the electric and magnetic vectors of radiation lie in the wave-front, it follows that in the two correlated systems, fixed and moving, the relative wave-fronts of radiation correspond, as also do the rays which are the paths of the radiant energy relative to the systems. The change of the time variable, in the comparison of radiations in the fixed and moving systems, involves the Doppler effect on the wave-length.

The Correlation between a stationary and a moving Medium, as regards trains of Radiation

113. Consider the aethereal displacement given by

$$(f_1,\ g_1,\ h_1)=(L,\ M,\ N)\,F\,(lx_1+my_1+nz_1-pt),$$

which belongs to a plane wave-train advancing, along the direction (l, m, n) with velocity V, or C/μ where μ is refractive index, equal to

$$p\,(l^2+m^2+n^2)^{-\frac{1}{2}},$$

in the material medium at rest referred to coordinates (x_1, y_1, z_1). In the corresponding wave-train relative to the same medium in motion specified by coordinates (x, y, z), and considered as shrunk in the above manner as a result of the motion, the vectors (f, g, h) and (a, b, c) satisfy the relation

$$\begin{aligned}\epsilon^{\frac{1}{2}}\left(\epsilon^{-\frac{1}{2}}f,\ g-\frac{v}{4\pi C^2}c,\ h+\frac{v}{4\pi C^2}b\right)&\\ =(L,\ M,\ N)\,F&\left\{l\epsilon^{\frac{1}{2}}x+my+nz-p\epsilon^{-\frac{1}{2}}\left(t-\frac{v}{C^2}\epsilon x\right)\right\}\\ =(L,\ M,\ N)\,F&\left\{\left(l\epsilon^{\frac{1}{2}}+\frac{pv}{C^2}\epsilon^{\frac{1}{2}}\right)x+my+nz-p\epsilon^{-\frac{1}{2}}t\right\}.\end{aligned}$$

As the wave-train in the medium at rest is one of transverse displacement, so that the vectors (f_1, g_1, h_1) and (a_1, b_1, c_1) are both in the wave-front, the same is therefore true for the vectors (f, g, h) and (a, b, c) in the correlative wave-train in the moving system, as was in fact to be anticipated from the

circuital quality of these vectors: the direction vector of the front of the latter train is proportional to $\left(l\epsilon^{\frac{1}{2}} + \frac{pv}{C^2}\epsilon^{\frac{1}{2}},\ m,\ n\right)$, and its velocity of propagation is

$$p\epsilon^{-\frac{1}{2}} \Big/ \left\{\left(l\epsilon^{\frac{1}{2}} + \frac{pv}{C'^2}\epsilon^{\frac{1}{2}}\right)^2 + m^2 + n^2\right\}^{\frac{1}{2}}.$$

Thus, when the wave-train is travelling with velocity V along the direction of translation of the material medium, that is along the axis of x so that m and n are null, the velocity of the train relative to the moving medium is

$$V\epsilon^{-1} \Big/ \left(1 + \frac{Vv}{C^2}\right),$$

which is, to the second order,

$$V\left(1 - \frac{v^2}{C^2}\right) \Big/ \left(1 + \frac{Vv}{C^2}\right) \text{ or } V - \frac{v}{\mu^2} - \left(\frac{1}{\mu} - \frac{1}{\mu^3}\right)\frac{v^2}{C}.$$

The second term in this expression is the Fresnel effect, and the remaining term is its second order correction on our hypothesis which includes Michelson's negative result.

In the general correlation, the wave-length in the train of radiation relative to the moving material system differs from that in the corresponding train in the same system at rest by the factor

$$\left(1 + 2l\frac{pv}{C^2}\right)^{-\frac{1}{2}}, \text{ or } 1 - lv/\mu C,$$

where l is the cosine of the inclination of the ray to the direction of v; it is thus shorter by a quantity of the first order, which represents the Doppler effect on wave-length because the period is the same up to that order.

When the wave-fronts relative to the moving medium are travelling in a direction making an angle θ', in the plane xy so that n is null, with the direction of motion of the medium, the velocity V' of the wave-train (of wave-length thus altered) relative to the medium is given by

$$\frac{\cos\theta'}{V'} = \frac{l\epsilon}{p} + \frac{v\epsilon}{C^2}, \quad \frac{\sin\theta'}{V'} = \frac{m\epsilon^{\frac{1}{2}}}{p},$$

where $(l^2 + m^2)/p^2 = V^{-2}$. Thus

$$\left(\frac{\epsilon^{-1}\cos\theta'}{V'} - \frac{v}{C^2}\right)^2 + \frac{\epsilon^{-1}\sin^2\theta'}{V'^2} = \frac{1}{V^2},$$

so that neglecting $(v/C)^3$,

$$V' = V - \frac{v}{\mu^2}\cos\theta' - \tfrac{1}{2}(1 - \mu^{-2})\,\frac{v^2}{\mu C}(1 + 2\cos^2\theta'),$$

where $\mu = C/V$, of which the last term is the general form of the second order correction to Fresnel's expression. In free aether, for which μ is unity, this formula represents the velocity relative to the moving axes of an unaltered wave-train, as it ought to do.

As (f, g, h) and (a, b, c) are in the same phase in the free transparent aether, when one of them is null so is the other: hence in any experimental arrangement, regions where there is no disturbance in the one system correspond to regions where there is no disturbance in the other. As optical measurements are usually made by the null method of adjusting the apparatus so that the disturbance vanishes, this result carries the general absence of effect of the Earth's motion in optical experiments, up to the second order of small quantities.

Influence of translatory motion on the Structure of a Molecule: the law of Conservation of Mass

114. As a simple illustration of the general molecular theory, let us consider the group formed of a pair of electrons of opposite signs describing steady circular orbits round each other in a position of rest**: we can assert from the correlation, that when this pair is moving through the aether with velocity v in a direction lying in the plane of their orbits, these orbits relative to the translatory motion will be flattened along the direction of v to ellipticity $1 - \frac{1}{2}v^2/C^2$, while there will be a first-order retardation of phase in each orbital motion when the electron is in front of the mean position combined with acceleration when behind it so that on the whole the period will be changed only in the second-order ratio $1 + \frac{1}{2}v^2/C^2$. The specification of the orbital modification produced by the

** The orbital velocities are in this illustration supposed so small that radiation is not important. Cf. §§ 151—6 *infra*.

translatory motion, for the general case when the direction of that motion is inclined to the plane of the orbit, may be made similarly: it can also be extended to an ideal molecule constituted of any orbital system of electrons however complex. But this statement implies that the nucleus of the electron is merely a singular point in the aether, that there is nothing involved in it of the nature of inertia foreign to the aether: it also implies that there are no forces between the electrons other than those that exist through the mediation of the aether as here defined, that is other than electric forces.

The circumstance that the changes of their free periods, arising from convection of the molecules through the aether, are of the second order in v/C, is of course vital for the theory of the spectroscopic measurement of celestial velocities in the line of sight. That conclusion would however still hold good if we imagined the molecule to have inertia and potential energy extraneous to (*i.e.* unconnected with) the aether of optical and electrical phenomena, *provided these properties are not affected by the uniform motion:* for the aethereal fields of the moving electric charges, free or constrained, existing in the molecule, will be symmetrical fore and aft and unaltered to the first order by the motion, and therefore a change of sign of the velocity of translation will not affect them, so that the periods of free vibration cannot involve the first power of this velocity.

115. The fact that uniform motion of the molecule through the aether does not disturb its constitution to the first order, nor the aethereal symmetry of the moving system fore and aft, shows that when steady motion is established the mean kinetic energy of the system consists of the internal energy of the molecule, which is the same as when it is at rest, together with the sum of the energies belonging to the motions of translation of its separate electrons. This is verified on reflecting that the disturbance in the aether is made up additively of those due to the internal motions of the electrons in the molecule and those due to their common velocity of translation. Thus in estimating the mean value of the volume-integral of the square of the aethereal disturbance, which is

the total kinetic energy, we shall have the integrated square of each of these disturbances separately, together with the integral of terms involving their product. Now one factor of this product is constant in time and symmetrical fore and aft as regards each electron, that factor namely which arises from the uniform translation; the other factor, arising from the orbital motions of the electrons, is oscillatory and symmetrical in front and rear of each orbit: thus the integrated product is by symmetry null. This establishes the result stated, that the kinetic energy of the moving molecule is made up of an internal energy, the same up to the first order of the ratio of its velocity to that of radiation as if it were at rest, and the energy of translation of its electrons. The coefficient of half the square of the velocity of translation in the latter part is therefore, up to that order, the measure of the inertia, or mass, of the molecule thus constituted. Hence when the square of the ratio of the velocity of translation of the molecule to that of radiation is neglected, its electric inertia is equal to the sum of those of the electrons which compose it; and the fundamental chemical law of the constancy of mass throughout molecular transformations is verified for that part of the mass (whether it be all of it or not) that is of electric origin.

116. Objection has been taken to the view that the whole of the inertia of a molecule is associated with electric action, on the ground that gravitation, which has presumably no relations with such action, is proportional to mass: it has been suggested that inertia and gravity may be different results of the same cause. Now the inertia is by definition the coefficient of half the square of the velocity in the expression for the translatory energy of the molecule: in the constitution of the molecule it is admitted, from electrolytic considerations, that electric forces or agencies prevail enormously over gravitative ones: it seems fair to conclude that of its energy the electric part prevails equally over the gravitative part: but this is simply asserting that inertia is mainly of electric, or rather of aethereal, origin. Moreover the increase of kinetic electric energy of an electron arising from its motion with velocity v depends on v^2/C^2, on the

coefficient of inertia of the aether, and on the dimensions of its nucleus, where C is the velocity of radiation: the increase of its gravitational energy would presumably in like manner depend on v^2/C'^2, where C' is the velocity of propagation of gravitation and is enormously greater than C. On neither ground does it appear likely that mass is to any considerable degree an attribute of gravitation.

The Transition from Electrons to Molecules

117. The main additional result derived from this second-order discussion is that if we assume all molecular forces to be electric forces, motion of a material system through the aether alters its dimensions in a minute but definite manner. A scrutiny, on all sides, of the basis of this inference is of course desirable. As a preliminary it is to be noticed that the molecular forces on the action of which it depends are extremely great in comparison with any distributions of force arising from finite currents or electrifications produced in the system as a whole. In the comparison between the two identical systems, one at rest the other in motion, of the analogy above developed, their electrons occupy corresponding positions in their spaces at all times: thus at first sight it is only systems in which the electrons are absolutely at rest that can be thus compared. But even in the case of dielectric bodies at rest, though the molecules are fixed the electrons are revolving in the molecules: yet that does not sensibly affect the application of the correspondence. For the only difference thereby introduced in it is that the phases of the orbital motions of those molecules of the moving material system that are situated further in advance, in the direction of the movement of the system, are slightly accelerated in comparison with the corresponding phases in the fixed system. Now the permanent or secular relations between molecules, supposed far enough apart not to interfere in a structural manner with each other so as to form compound molecules, are independent of these relative phases: to obtain them we in fact replace each molecule by its steady secular equivalent in the Gaussian sense, as has to be

done in a representation of their magnetism, and thus the phase-change makes no difference for the present purpose. The case is however different when there are electric currents flowing in the system, for that involves the transfer of some electrons into entirely new positions, it may be at a finite distance: these wandering electrons or ions interfere with the exact statement of the correlation, and they interfere to a like extent with any conclusions that may be drawn from it, as to change of form of solid bodies carrying currents arising from their motion with the Earth through the aether.

How far then is the correlation between the fixed material system and the moving system modified by electric conduction? In the theorem the position of each electron in the material medium in motion, at time t, corresponds with that which it would occupy in the medium at rest at time $t - vx/C^2$. When the material medium is a solid dielectric mass, the mean position of the electron is the same at all times, and as we have seen this element of time does not enter into the comparison at all: but when the medium is conducting, the electric currents in it involve migration of electrons through it, and we must consider how far the correspondence is thereby prejudiced. Only two views of the nature of conduction, in this connexion, are open. The current in metals may possibly (but not likely) be carried by very few electrons, in which case they will migrate with sensible speed; but the smallness of their number, compared with the total number of combined electrons, prevents their changes of position from sensibly affecting the molecular structure of the medium: we know in fact that the mechanical structure of a conductor is not sensibly affected when it carries a current. On the other hand a considerable proportion of the electrons may take part in carrying the current; in which case their velocity of migration is excessively minute, as for instance follows from the phenomena of migration in electrolysis*; and the discrepancy of position of those electrons, in the application of the correlation theorem, involving the factor v/C^2 as well as this velocity, is negligible to an order higher than the second, just as was the discrepancy of phase in the individual molecular

* Cf. Appendix ~~B~~ C, § 6.

orbits. To reach this conclusion, it is by no means necessary to assume that we have any knowledge of the process by which ionisation, or the passing on of electrons from molecule to molecule, occurs in conductive processes.

Influence of Convection on Conductivity

118. In this connexion we can gain some knowledge of the nature and amount of the effect of the Earth's motion on electrolytic conduction. If the convective velocity v is in the direction of the current, and the actions between the ions are, as usual in electrolytic theory, assumed to be wholly electric, and w and w' represent velocities of positive and negative ions, then the position of the positive ion in the electrolyte at rest is given by $x = wt$; hence (§ 112) in the electrolyte in motion with the same electric force it is given by $x = w\left(t - \frac{v}{C^2}x\right)$, so that $x = \frac{w}{1 + vw/C^2}t$; thus the velocity of the positive ion relative to the moving electrolyte is $w\Big/\left(1 + \frac{vw}{C^2}\right)$. The velocity of the negative ion is similarly $w'\Big/\left(1 - \frac{vw'}{C^2}\right)$. The electric current, being determined by the sum of these velocities, is altered as regards these ions in the ratio of $w + w' - \frac{v}{C^2}(w^2 - w'^2)$ to $w + w'$ approximately; it is thus diminished in the ratio $1 - v(w - w')/C^2$; and the conductivity of the electrolyte is diminished in this ratio, where now $w - w'$ represents an average value, the difference of the velocities of drift of positive and negative ions. This change of conductivity is a unilateral one, being reversed when the direction of the current is reversed: it is at most of the second order of small quantities: it vanishes altogether, or rather becomes of two orders higher, when the velocities of the positive and negative ions are the same. It may be remarked incidentally that, as the numbers of positive and negative ions taking part in the current of conduction are the same, the specification of that current with reference to moving matter is just the same as with reference to the stationary aether.

The Argument of Lorentz regarding the Michelson experiment

119. As an assistance to the formation of a judgment on these questions, it will be convenient to insert here a free translation of the considerations by which Lorentz* supported the possibility of an explanation, of the kind above developed, of the negative result of Michelson's experiments on the influence of material convection on phenomena of optical interference.

"However extraordinary this hypothesis may appear at first sight, it must be admitted that it is by no means gratuitous, if we assume that the intermolecular forces act through the mediation of the aether in a manner similar to that which we know to be the case in regard to electric and magnetic forces. If that is so, the translation of the matter will most likely alter the action between two molecules or atoms in a manner similar to that in which it alters the attraction or repulsion between electrically charged particles. As then the form and the dimensions of a solid body are determined in the last resort by the intensity of the molecular forces, an alteration of the dimensions cannot well be left out of consideration.

"In its theoretical aspect there is thus nothing to be urged against the hypothesis. As regards its experimental aspect we at once notice that the elongation or contraction which it implies is extraordinarily minute. It would involve a shortening in the diameter of the Earth of about 6½ centimetres. The only experimental arrangements in which it could come into evidence would be just of the type of this one of Michelson's which first suggested it.

"It is worthy of remark, that we are led precisely to this law of alteration of dimensions when we assume *first* that, without taking account of molecular motions, in a solid body left to itself the forces of attraction and repulsion acting on each molecule maintain themselves in equilibrium, and *secondly* —for which there is admittedly no evidence—that the same law applies to these molecular forces, as regards their alteration

* 'Versuch einer Theorie...' 1895, §§ 91—2.

by convection, that has been demonstrated for the electrostatic attractions of moving charges. Let us understand by S_1 and S_2, not as previously two systems of charged particles, but two systems of molecules,—the second at rest and the first in motion with velocity v in the direction of the axes of x,—between whose dimensions the previously given relation holds; then since in both systems the x components of the forces are the same, while the y and z components differ by the factors given, it is clear that the forces in S_1 will balance when that is the case for those of S_2. If therefore S_2 is the state of equilibrium of a solid body at rest, the molecules in S_1 have just those positions in which they could subsist under the influence of the motion of translation. The displacement into this new configuration would therefore take place of itself, involving a contraction in the direction of motion in the ratio of unity to $(1 - v^2/C^2)^{\frac{1}{2}}$.

"In reality the molecules of a body are not at rest, but corresponding to each position of equilibrium they are in a state of stationary motion. How far this difference is of importance for the phenomena treated, must be left undetermined: the experiments of Michelson and Morley leave for it a comparatively wide range of effect on account of the unavoidable errors of observation."

The force of the last remark is removed by Michelson's more recent observations* with a longer ray-path, in which the delicacy was so great that it was necessary for consistent results to get rid of the air; even then no trace of uncompensated effect was observed.

Are the linear equations of the Aether exact?

120. In favour of the view that the interactions between atoms are in very great part those necessitated by the aether whose properties are revealed in electric and optical phenomena, there is, in addition to the inherent theoretical difficulty in conceiving any other kind of interaction, the actual fact that on the lines of the above argument such a view does account for a definite and well-ascertained experimental result, that of

* *American Journal of Science*, 1897.

Michelson, above discussed, which has hitherto stood by itself as the only quantitative observational evidence that has a bearing on this question. It can be said on the other side that this view of aethereal action does not directly cover gravitational phenomena, unless the rather artificial pulsatory theory of gravity is allowed*. But there is another aspect of the matter. The equations of the free aether, as revealed by MacCullagh's optical analysis, are linear equations: they in fact must be so if all kinds of radiations are to travel with the same speed in the celestial spaces. In Maxwell's hands, equivalent relations with the appropriate generalization were arrived at on the electric side, and formed a basis for the explanation of the whole *plexus* of electrodynamic and optical phenomena. Further theoretical discussion has in all directions tended to widen the scope and enhance the inherent simplicity of this scheme. The question arises whether there is anything to gainsay a view that this simple linear scheme is only the first approximation, a very close one however, to an analytical specification of the aether: just as the linear scheme of equations of the theory of propagation of sound covers the whole of the phenomena of acoustics, although in arriving at those equations from the dynamics of the atmosphere all terms involving the square of the ratio of the velocity of the actual aereal disturbance to the velocity of its propagation are neglected, for the reason that their consequences are outside the limits of observation in that domain. Why then should not relatively minute phenomena like gravitation be involved in similar non-linear terms, or terms involving differentials of higher orders, in the analytical specification of the free aether, which are as insignificant compared with the main fully ascertained linear terms as is the gravitation between two electric systems compared with their mutual electric forces? Against this there is a subjective reluctance to disturb the ideal simplicity of the aethereal scheme: but there is no help for that if its content is not sufficiently extensive for the facts. Of more weight is the circumstance that a train of radiation from a distant star would change its form as it advanced across

* Cf. *Phil. Trans.* 1897 A, p. 217.

space, that there would in fact be optical dispersion in the free aether if such second-order terms existed. The amount of such dispersion that would be at all allowable is known to be excessively minute, from the circumstance that celestial bodies on emerging from eclipse or occultation show no changes of colour: the smallness of the amount that would be required may be estimated by comparing the electric force between two ions with their gravitational attraction. Unless the effects of such terms of higher order, in the equations of aethereal activity, increased enormously in importance at molecular distances, relatively to the main linear terms, the proposition that the interactions of molecules are mainly of electric quality would remain valid: now such increase of importance does not seem likely as regards the mechanism of gravitation, for gravity and electric force both obey the same law of the inverse square of the distance, a law which in fact belongs, of mathematical necessity, to the steady permanent interactions between any kinds of molecular nuclei of elastic or motional disturbance in an extended medium, which are of the type of simple poles.

A question of some interest arises, as to whether the assumption that the linear equations of free aether are a first approximation, obtained by the omission of non-linear terms, would imply a virtual recognition of structure in that medium. A presumption of this kind would be useless except for purposes of vivid illustration after the manner of mechanical models, so long as there is absolutely no means of experimenting on the properties of free aether: and this practically comes to the same thing as taking such structure to be non-existent.

121. There is thus little to be urged in favour of leaving this loophole for the explanation of gravitation. On the other side moreover there appears to be the fatal objection that any action accounted for in this way would have relations with radiation, including a velocity of transmission of the same order as that of light. The knowledge that the speed of transmission of gravitation, if finite at all, enormously transcends that of radiation, shows that it forms no objection to a theory of electric and radiant phenomena that gravitation is not found

to be involved in it. An analogy in fact suggests itself with the molecular electric theory as developed by Weber, Kirchhoff and their school, which gave a complete account of ordinary material electric phenomena, and only failed when the totally different region of radiation came into the discussion. It seems fair to conclude, in the one case as in the other, that in the constitution of the energy-relations on which the phenomena depend, a new property of the medium becomes explicitly involved in the more refined theory (not merely implicitly as in the energy-function that suffices for ordinary material electrodynamics) such for instance as the incompressibility that is utilized in the pulsatory theory or illustration of gravitation. The general reasons against the notion that the fundamental property of mass in matter is in direct connexion with the mechanism by which gravitation is transmitted have been given above (§ 116). There appears then, as yet, to be nothing to tempt us to depart from the natural prepossession, by considering the simple linear equations of the aether to be other than exact.

Dimensional Relations: in connexion with the definite scale of magnitude of Atomic Structure

122. Important considerations bearing on the question as to how far atoms of matter are constituted simply of singularities in the aether, practically point-nuclei, may be derived from the Newtonian principle of dynamical similarity, as utilized above (§ 112). Let us compare two such aethereal systems represented one by ordinary the other by subscripted variables, between which there is a correspondence given by

$$(x,\ y,\ z) = k\,(x_1,\ y_1,\ z_1),\quad t = lt_1,$$
$$(a,\ b,\ c) = \vartheta\,(a_1,\ b_1,\ c_1),\quad (f,\ g,\ h) = \phi\,(f_1,\ g_1,\ h_1);$$

the aethereal equations for the one system will be identical with the aethereal equations for the other provided

$$\vartheta/k = \phi/l,\quad \phi/k = \vartheta/l,$$

so that

$$\vartheta = \phi \text{ and } k = l.$$

Hence, given any one existing system of electrons with point-nuclei, another system is possible in the same aether having all distances and times reduced in any the same ratio, and electric displacement and magnetic flux independently reduced in any other the same ratio. But if the electrons of this correlated system are to be of the same strengths as the original ones ϕ/k^2 must be unity; hence the scale must be altered in the same ratio throughout, as regards length, time, ~~and~~ throughout the inductions. Thus, given any existing steady system of electrons, the same system altered to any other scale of linear magnitude is possible if there are none but electric actions. This is on the hypothesis which is here generally adopted, that the dimensions of the nucleus of an electron are so small, compared with the mutual distances of electrons, that these dimensions are not sensibly involved in the forces between them. If this condition is left out the constancy of volume of the nucleus will have to be taken into consideration in the dimensional transformation, so that k must be unity; and this indefiniteness of linear scale in a material body cannot exist. The size of a molecule would also be rendered determinate if residual non-linear terms in the aethereal equations became sensible at intermolecular distances. Thus, these saving hypotheses being excluded, if the atoms of matter were constituted electrically, and the forces between them were wholly of electric origin, there would be nothing to determine the scale of an isolated system as regards time and space: and different systems need not be always of the same scale of magnitude as regards their atomic structure.

123. A similar deficiency of definite scale would also be expected to exist in any hydrodynamical theory or illustration which would construct an atom out of vortex rings. Thus let us consider a system of vortex rings, (ξ, η, ζ) being the vorticity at the point (x, y, z), and compare with another system in another space (x', y', z') such that the coordinates of corresponding points are connected by the relation

$$(x', y', z') = k(x, y, z),$$

while the vorticities at these points are connected by the relation

$$(\xi', \eta', \zeta') = \kappa(\xi, \eta, \zeta).$$

The formula of von Helmholtz for the velocity (u, v, w) of the fluid in terms of the vorticity, being of type

$$u = \int \left(\eta \frac{d}{dz} - \zeta \frac{d}{dy}\right) \frac{1}{r} \, d\tau,$$

gives $$(u', v', w') = k\kappa (u, v, w).$$

But the systems will maintain their correspondence of configuration throughout succeeding time, only provided always

$$(u', v', w') = k (u, v, w);$$

hence $\kappa = 1$ while k is arbitrary. Thus if any vortex-system is compared with another one expanded as regards linear scale k times, and the vorticity is at each point unaltered, so that the circulations of the vortices in the new system are all increased k^2 times, then their subsequent histories will correspond exactly.

The circulation of the vortex is however in the dynamical theory an unalterable constant, so that the one system cannot be changed by natural processes into the other. Let us try therefore to avoid this difference by a change of the time scale as well, so that $t' = \lambda t$; then for continued correspondence

$$(u', v', w') = k\lambda^{-1} (u, v, w):$$

hence $k\kappa = k\lambda^{-1}$, so that $\kappa = \lambda^{-1}$; and the strengths of the vortices are altered in the ratio $k^2\lambda^{-1}$, which must be a constant. Thus if the scale of time is increased λ times, and that of linear magnitude $\lambda^{-\frac{1}{2}}$ times, and the corresponding vortex filaments are of the same strengths, the systems will continue permanently in correspondence. This is however on the assumption that the vorticity is around a vacuous core, or a fluid core so thin that its actual section does not affect the mutual actions of the vortices: for the change of linear scale will alter the volume of the core of each ring. There is under these conditions nothing in the hydrodynamical forces to fix the scale of magnitude of an isolated vortex-system with

vacuous cores; the same system can equally exist with linear dimensions k times as great when all the time constants will be diminished k^2 times.

124. The definiteness of scale of the molecules of material systems thus precludes the possibility of their being constituted of singularities of a uniform continuum, of either of these kinds with nuclei undistinguishable from mathematical points. The constancy of inertia and gravity throughout all chemical transformations forms practically sufficient evidence for the physicist that all matter is built up out of the same primordial stuff: this stuff, if it is constituted of intrinsic singularities in a uniform aethereal continuum with relations exactly linear, must thus be made up of elements of type rather more complex than simple positional and motional singularities with nuclei devoid of sensible volume. Another element apart from finiteness of dimensions of nuclear structure that could enter, on the theory of an aether exactly linear in its relations, is that of time: for example it has been seen how the gravitation of atoms can be imitated by supposing a definite periodic time of pulsation to be associated with each electron. A change of scale such as that above discussed would then change the forces of gravitation, unless possibly the time of pulsation could be suitably altered and the change thus counteracted.

125. In the above considerations there is strong evidence that gravitation is not to be expected to be appreciably involved within the scheme which suffices to cover the phenomena of electrodynamics and optics. The introduction of the time-relation inherent in pulsating nuclei seems still to be the only obvious way of representing it, in default of its arising from second-order terms in the dynamical relations of the aether. The permanence of scale of magnitude of the material atoms of various types involves the presence of actions depending on the magnitude and structure of the electric nuclei, which though they may be purely aethereal are local, and thus not pertinent to general electrical and optical theory: the existence

of a configuration of minimum energy in the molecule in fact implies finite structure in the nuclei in some such way. At the same time the Michelson interference-result indicates that these other agencies play a quite subordinate part in our present problems: for the correlation above established, which involves that result, only holds strictly for electrons whose nuclei are considered as mere points in comparison with their mutual distances. Atomic inertia other than that which comes from the aether in some way it seems impossible to conceive: but in other respects we are hardly on the threshold of the structure of the atom. The problem there involved is not to assign a structure so minutely definite that it will include the whole complex of chemical actions, but rather to ascertain how much must be postulated in order to correlate the main features of those universal agencies, affecting all kinds of matter, with which the theoretical side of physical science deals.

SECTION IV

CHAPTER XII

ON OPTICAL ROTATIONS MAGNETIC AND STRUCTURAL

126. THE rotation of the plane of polarization of light, whether by naturally active media, or under the influence of magnetism induced in ordinary media, is dynamically a secondary and subordinate phenomenon. But in the testing of theories regarding the interaction of aether and matter, particularly in questions relating to velocity of propagation, it can take an important part. The ordinary mode of determining, by means of interference bands, how much one wave-train has outrun another proceeds by counting wave-lengths, only considerable fractions of a wave-length being recognizable. But in the interference of circularly polarized waves the single wave-length is so to speak spaced round a circle, and the delicacy of the measurement is limited only by the angular fraction of the circumference to which the instrumental graduations can be set with precision in order to obtain extinction of the light: thus an extremely minute alteration in the velocity of a circular wave can be recognized. The change of circular waves into elliptic ones, on reflexion, however practically limits this method of observing interference to the phenomena of media which rotate the plane of polarization.

As preliminary to an investigation of the interaction of optical rotation with the Earth's motion through space, we

proceed to a review of its general character on the lines of the present theory.

In attempting to treat the optical relations in a material substance, considered as a single modified medium instead of as simple aether under the reaction of material molecules, the only mode of representation of magneto-optic phenomena that lay open was* by addition of a subsidiary mixed term to the energy function, so as to express a connexion between the optical waves and the magnetic field. The working out of that hypothesis into the theory of reflexion necessitates the introduction of an electromotive pressure in the incompressible aether†, which is a type of stress not excited at all in ordinary refraction.

The method of taking into consideration the influence of the separate imbedded molecules, which has formed the basis of the present discussion, puts us in a position to scrutinize the ultimate validity of that type of abstract formulation of the problem. A molecular investigation of this kind is in fact also called for on other grounds, in so far as physico-chemical experiment has indicated the existence of molecular equivalents in both the magnetic and the structural kinds of optical rotation. It will suffice to consider radiation of one definite period: the effect of dispersion on the rotation will be obtained by simply changing to a new period and to the corresponding new optical constants, because the interaction of the minute rotational property with ordinary dispersion is negligible. We thus have to deal with electric and magnetic force and electric and magnetic flux, such that each flux is derived from the other force by the universally valid circuital relations; while the influence of the molecules of the ponderable medium will as usual impress itself only on the form of the relations, depending on the constitution of the medium, which connect each flux with the corresponding

* Maxwell, 'Treatise', § 824.

† This refers to Maxwell's type of energy-term, which is of the quadratic character that would naturally be assumed: Mr Basset has shown that a form of term can be specified, involving the continued product of the imposed magnetic field, the electric polarization, and its time-gradient, which will lead to the equations of the theory described below.

force. In light-waves we can safely take the magnetic permeability to be unity; so that there remains at our disposal, for modification in rotational manner, only the form of the relation between the total circuital electric displacement (f'', g'', h''), equal to $(f', g', h') + (4\pi c^2)^{-1}(P, Q, R)$, and the electric force (P, Q, R). The relation between the induced polarization (f', g', h') of the molecules, and the electric force, would under ordinary circumstances be a simple linear one, which must be self-conjugate however aeolotropic the medium may be, as Lord Kelvin showed, in order to avoid perpetual motions. Under circumstances of optical rotation, the law of rotatory dispersion inversely as the square of the wave-length, verified by Biot and by Verdet, easily shows that the rotatory terms in the equations of propagation in the medium must be of the third order in the differential coefficients; and this requires that the polarization shall be a linear function of the first differential coefficients of the inducing electric force, as well as of that force itself. For the case of the structural rotatory property of quartz and other substances these differentiations will naturally be spacial: in magnetic rotation various considerations* show that they must be with respect to time. The condition has still to be introduced that these linear relations between flux and force, thus extended to include differential coefficients of the vectors concerned, are consistent with an energy function, and so avoid the possibility of perpetual motions.

127. Let us consider first the case of electric polarization induced in a body situated in a magnetic field. The energy of the distribution of polarization (f', g', h') established by the electric force (P, Q, R) must be $\frac{1}{2}\int(Pf' + Qg' + Rh')\,d\tau$, and is thus, per unit volume, a quadratic function of (P, Q, R) and to a minute extent of $d/dt\,(P, Q, R)$, the rotatory property coming in through the latter part. It must therefore be of the form

$$\int\left\{F_2(P, Q, R) + a_{11}P\frac{dP}{dt} + \ldots + a_{12}P\frac{dQ}{dt} + a_{21}Q\frac{dP}{dt} + \ldots\right\}d\tau,$$

* Cf. British Association Report, 1893, 'On the influence of Magnetism on Light.'

where $F_2(P, Q, R)$ is a quadratic function equal in the case of an isotropic medium to $(K-1)/8\pi C^2 . (P^2+Q^2+R^2)$. The variation of this volume-integral must, by the definition of (P, Q, R) as the force producing change of polarization, be equal to

$$\int (P\delta f' + Q\delta g' + R\delta h')\, d\tau;$$

but it is $\frac{1}{2}\delta \int (Pf' + Qg' + Rh')\, d\tau$: hence it is also expressible in terms of (P, Q, R) as independent variable, in the form

$$\int (f'\delta P + g'\delta Q + h'\delta R)\, d\tau.$$

This expression must be identical with the result of direct variation of the energy expressed in terms of (P, Q, R), except as regards terms at the time-limits, arising from partial integration, which are inoperative in the formation of dynamical equations. We thus obtain the relations

$$f' = \frac{dF_2}{dP} + \frac{a_3}{4\pi C^2}\frac{dQ}{dt} - \frac{a_2}{4\pi C^2}\frac{dR}{dt},$$

$$g' = \frac{dF_2}{dQ} + \frac{a_1}{4\pi C^2}\frac{dR}{dt} - \frac{a_3}{4\pi C^2}\frac{dP}{dt},$$

$$h' = \frac{dF_2}{dR} + \frac{a_2}{4\pi C^2}\frac{dP}{dt} - \frac{a_1}{4\pi C^2}\frac{dQ}{dt},$$

where

$$(a_1, a_2, a_3)/4\pi C^2 = (a_{23} - a_{32},\ a_{31} - a_{13},\ a_{12} - a_{21}).$$

The effect on the material medium, of the extraneous magnetic field or other vector agency, is thus to modify the induced electric polarization, by addition of a part at right angles to $d/dt\,(P, Q, R)$ and to the vector $(a_1, a_2, a_3)/4\pi C^2$, and equal to their vector product. But the question also arises whether the ordinary dielectric coefficients, those namely of the function $F_2(P, Q, R)$, are sensibly altered by the imposed magnetic field. This point can be settled as usual by aid of the principle of reversal (§ 88). When the electric force and the imposed

magnetic field and the time are all reversed, the effect on the induced electric polarity must be simple reversal: hence a reversal of the magnetic field cannot affect the coefficients in $F_2(P, Q, R)$: hence any changes in these coefficients must depend on the square or other even powers of the imposed magnetic force: but the rotational terms depending on the first power of this force are known to be very small, therefore any terms depending on its second power are wholly negligible**. This conclusion has been fully verified in an experimental investigation by Mascart, who has found that the mean of the velocities of a right-handed and a left-handed circular wave-train is equal to the velocity proper to the medium when removed from the influence of magnetic force.

The general result is noteworthy, that even in a crystalline medium any dependence, from whatever cause, of electric polarization on the time-rate of change of the inducing electric force, must consist in the addition of a purely rotational part isotropic around an axis.

128. When this relation between electric polarization and electric force is substituted in the electrodynamic circuital equations of types

$$4\pi\left(\frac{df'}{dt}+\frac{1}{4\pi C^2}\frac{dP}{dt}+\sigma P\right)=\frac{d\gamma}{dy}-\frac{d\beta}{dz},$$

$$-\frac{d\alpha}{dt}=\frac{dR}{dy}-\frac{dQ}{dz},$$

the equations of magneto-optic propagation will be obtained. When P, Q, R are chosen as independent variables, these equations of propagation are of type

$$C^2\nabla^2 P=\frac{d^2}{dt^2}\left(K'P+a_3\frac{dQ}{dt}-a_2\frac{dR}{dt}\right),$$

where

$$K'=K+4\pi\sigma C^{+2}(d/dt)^{-1};$$

and the surface conditions in the problem of reflexion of

** More precisely, the effect is of the order of the ratio of the forces exerted by the imposed magnetic field on the electrons in the molecule to their own mutual forces: this ratio must thus be very small and its square negligible.

radiation are that the tangential components of the electric force (P, Q, R), and of its curl which determines the magnetic force, shall be continuous. Let us apply these equations of propagation to a wave travelling along the axis of z; they then become

$$c^2 \frac{d^2P}{dz^2} = K' \frac{d^2P}{dt^2} + a_3 \frac{d^3Q}{dt^3},$$

$$c^2 \frac{d^2Q}{dz^2} = K' \frac{d^2Q}{dt^2} - a_3 \frac{d^3P}{dt^3},$$

which are practically equivalent, a_3 being very minute, to the form of Maxwell and Verdet,

$$c^2 \frac{d^2P}{dz^2} = K' \frac{d^2P}{dt^2} + a_3 c^2 K'^{-1} \frac{d^3Q}{dz^2 dt},$$

$$c^2 \frac{d^2Q}{dz^2} = K' \frac{d^2Q}{dt^2} - a_3 c^2 K'^{-1} \frac{d^3P}{dz^2 dt}.$$

The previous form, which is slightly more convenient, may be condensed in the case of a transparent medium by the use of a complex variable into the single equation

$$c^2 \frac{d^2}{dz^2}(P + \iota Q) = \left(K \mp \iota a_3 \frac{d}{dt}\right) \frac{d^2}{dt^2} (P + \iota Q),$$

showing immediately that all waves of permanent type are circularly polarized, right and left-handed ones of period $2\pi/p$ travelling with the different velocities $c(K \pm a_3 p)^{-\frac{1}{2}}$. In traversing a thickness l of the medium the one gains on the other by $la_3 p/cK^{\frac{1}{2}}$ in time, or by $4\pi^2 l c a_3/K^{\frac{1}{2}}\lambda^2$ in phase, λ being the wave-length in vacuum. It is thus the quantity $a_3/K^{\frac{1}{2}}$, or a_3/μ where μ is refractive index, that is usually taken in chemical physics as the measure of the rotatory power of the material medium.

129. We proceed to examine how far the principles which lead to Lorentz's molecular refraction-equivalent for transparent media* are applicable to the investigation of a molecular rotation-equivalent. If n denote the number of molecules, all of them rotationally active, per unit volume, the equations

* Cf. *Phil. Trans.* 1897 A, p. 232.

connecting the induced polarization in the molecules with the force which induces it must be of type

(m', n') is direction of the [illegible]

$$f' = (P + \tfrac{4}{3}\pi C^2 f')\, n\epsilon + n\eta\, n'\frac{dQ_1}{dt} - n\eta\, m'\frac{dR_1}{dt},$$

where ϵ and η are molecular constants, the latter proportional to the magnetic field. It is here assumed that the force (P_1, Q_1, R_1) polarizing each molecule is equal to that near the centre of a spherical cavity with the molecule situated inside it, so that

$$Q_1 = Q + \tfrac{4}{3}\pi C^2 g' = \tfrac{1}{3}(K+2)\,Q$$

approximately, and similarly for R_1. Thus

$$(1 - \tfrac{4}{3}\pi n C^2 \epsilon) f' = n\epsilon P + \tfrac{1}{3}(K+2)\, n\eta\, n'\frac{dQ}{dt} - \tfrac{1}{3}(K+2)\, n\eta\, m'\frac{dR}{dt};$$

where by the definition of the dielectric constant

$$K - 1 = 4\pi n C^2 \epsilon / (1 - \tfrac{4}{3}\pi n C^2 \epsilon),$$

so that $(K-1)/(K+2)$ is equal to $\tfrac{4}{3}\pi n C^2 \epsilon$ and is therefore proportional to the density, in accordance with Lorentz's law of refraction-equivalents. Hence finally we have for the total electric displacement (f'', g'', h'') equations of the type

$$f'' = \frac{K}{4\pi C^2} P + \tfrac{1}{9}(K+2)^2\, n\eta\, n'\frac{dQ}{dt} - \tfrac{1}{9}(K+2)^2\, n\eta\, m'\frac{dR}{dt}.$$

The specific rotation r per unit length of a transparent medium is thus $(K+2)^2 n\eta / 9K^{\frac{1}{2}}$; so that K being μ^2, the rotation characteristic of each molecule is $(\mu^2+2)^2\eta/9\mu$; and on this analysis $\mu r/(\mu^2+2)^2\rho$, where ρ is the density, not r/ρ itself, should be an additive physico-chemical constant on the analogy of for example specific heat. If we apply Lorentz's law of specific refractive power, verified just above, that $(\mu^2-1)/(\mu^2+2)$ is proportional to the density, we find that for the same pure active medium under different circumstances r should be proportional to $(\mu^2-1)(\mu^2+2)/\mu$*. The experi-

* For the case of solutions sufficiently dilute, so that the index $K^{\frac{1}{2}}$ is practically constant, the specific rotation per active molecule in unit volume is of course constant: for different neutral solvents an argument similar to the above shows that it should vary as $(\mu^2+2)/\mu$ where μ is this index.

mental examination of this subject has been effected chiefly by H. Becquerel from the physical side and by W. H. Perkin from the chemical side: the former has advanced an empirical relation, r proportional to $(\mu^2-1)\mu^2$, as in a rough way representing in many cases the influence of change of density, but it would seem that no rational relation has been found. We might be tempted to explain the absence of a definite law by the hypothesis, whose equivalent has been suggested by Verdet, that the relation between electric polarization and electric force involves also rotational terms with fluxions of higher odd order with respect to the time than the first fluxion to which they have been here confined: it is only in problems involving dispersion that these terms will make any difference in the theory, but additional terms of magnitude sufficient to be of any use for the present purpose would wholly upset Verdet's experimental result that the rotation is roughly as the inverse square of the wave-length.

See p. 354.

130. According to our present view a molecule is, or at least involves, a collocation of electrons revolving round each other in stable orbits: the electric force of the field pulls the electrons different ways, thus disturbing the configuration of their steady orbits so as to introduce effective electric polarity: the influence of the magnetic force of the field is complex, as it tends to orientate these orbits as a whole without change of their dimensions, thereby introducing paramagnetic polarity, while it also tends to alter their forms by contracting the projections of their areas in the plane at right angles to its direction thus introducing polarity of the opposite or diamagnetic kind**. These various actions involve energy terms for each individual molecule, and the sum for all the molecules, if it could be formed, would represent the total energy of the disturbance of the medium. But such a mere aggregate of

** If the electrons were all of the same sign and subject to a central attraction, there would be no orientation but only a permanent rotation of the orbital system around the axis of the imposed magnetic field, the effect being on the whole paramagnetic or diamagnetic according as the electrons are positive or negative. Cf. *Phil. Mag.* Dec. 1897.

terms would be of no use for applications to matter in bulk: what we are concerned with there is the mechanical part of the energy, which must be an analytical function of the specification of matter by volume, determined as to mathematical form by the character of the molecular actions, but with coefficients whose values are to be obtained only by direct experiment. For each molecule the axis of the induced effect will usually be different from that of the inducing force; so that, when the molecules are all naturally orientated as in crystals, the relations for the medium which they constitute will show crystalline as well as rotational quality.

131. The physical explanation of the magnetic rotational property has already been indicated (§ 91). A circularly polarized beam in which the direction of rotation is right-handed will have a relation to the revolving electrons of the molecules, as orientated by the magnetic field, different from that of a left-handed beam, and will therefore pass across the medium with different velocity. Each electron, as it is moved by the aethereal displacement belonging to the radiation, resists with its own definite inertia; so that the circumstances are of similar general type to those of the propagation of circularly polarized waves in a material medium endowed with intrinsic angular momentum, for which the same form of equations is known to apply*. Conversely the reaction exerted by the disturbed aether on the molecule in the magnetic field will be different according as the disturbance arises from one or the other kind of circularly polarized beam: the forced periods of the molecule will therefore be different and in consequence also the absorption, in the two cases. The periods of any dynamical system vibrating about a configuration of rest, and

* *Proc. Lond. Math. Soc.* xxiii. 1891, p. 127. The comparison in the text does not imply that the two problems are analogous in detail: in fact the electromagnetic reaction of the revolving electrons to aethereal waves is a wholly different thing from the reaction of their inertia to waves of material displacement. The magnetic axis of a molecule is *not* to be identified with an axis of resultant material angular momentum: if the electrons contained in it were all of the same sign this would be more or less the case, but as things are, positive and negative electrons going the same way round give the same sign for material angular momentum while they give different signs for magnetic moment.

also of one in which gyrostatic influence is wholly dominant, are stationary so that ordinary slight disturbance of the structure of the system itself does not alter them except to the second order of small quantities†: it is the extraneous character of the disturbance that is here effective as regards its first power. Each absorption line, say of sodium vapour in a magnetic field, will thus be more or less widened, and its mean position also slightly shifted but only to a higher order of small quantities: and the same will apply to each line in the emission spectrum*. It might be in part alteration of the capacity of the molecule for electric polarization arising from structural change due to the magnetic field, and in part this change in its free periods acting in the usual dispersive manner, that alters the velocity of propagation of circularly polarized light and so produces the Faraday effect. The connexion has been illustrated by G. F. FitzGerald§ by a special calculation for solitary electrons describing circular paths under central attraction, in which the Faraday effect is ascribed wholly to the alteration of molecular periods represented by that of Zeeman. This finds the origin of the rotational term that exists in the relation connecting induced polarization and electric force, when the medium is under the influence of an extraneous magnetic field, wholly in the Zeeman change of molecular periods, which is in keeping with the circumstance that the rotational term involves time-differentiation. Even in a general type of molecule changes of the orientations and configurations of the orbits of the electrons arising from the magnetic field could hardly have an influence as well as changes of their periods**: for such an influence would be structural, and therefore by Lord Kelvin's application of the perpetual motion axiom (§ 127) it could not be rotational.

† Rayleigh, 'Theory of Sound,' § 90.

* The experiments of Zeeman and others, announced since this was first written, have shown that the actual relations are more extensive and definite, and more complex, than those above foreshadowed.

§ *Roy. Soc. Proc.* 1898.

** The subject can be treated on a more definite basis: cf. a communication to *Camb. Phil. Soc.* Mar. 6, 1899, in 'Nature,' April 20, in connexion with *Phil. Mag.* Dec. 1897: also Appendix F.

132. The conditions to be satisfied at the interface separating two media do not on the present theory require the introduction of an electromotive pressure into the equations of magneto-optic reflexion: for the continuity of the tangential magnetic force secures that of the normal electric flux, and *vice versâ*, by the nature of the fundamental circuital relations. The intrinsic reason why such a pressure is avoided is that each molecule is taken to affect the aethereal vibrations individually, either wholly statically, or in addition (when dispersion is included) by synchronous vibration of the dynamical system forming the single molecule by itself, but not of a system formed by the *plexus* of molecules bound together to an appreciable extent by mutual constraints: thus an electromotive pressure could have no meaning. If the molecules were connected in this way, the propagation of the transverse optical wave in the aether would be accompanied by that of another wave from molecule to molecule, and the whole scheme of equations of optical propagation in material media would in so far be affected: it is in fact a very minute longitudinal wave of this kind going at practically infinite speed that is represented by the pressural term in the theory of reflexion in a magnetic field, which is necessitated by Maxwell's and FitzGerald's mode of formulation of the problem.

The actual problem of magneto-optic reflexion is concerned mainly with the application to metallic media. Assuming the above relation between polarization and force, of type

sign of (a_1, a_2, a_3) here changed

$$4\pi C^2 f' = (K-1)P - a_3 \frac{dQ}{dt} + a_2 \frac{dR}{dt},$$

or what is practically the same,

$$P = \frac{4\pi C^2}{K-1} f' + a_3' \frac{dg'}{dt} - a_2' \frac{dh'}{dt},$$

where

$$a'/a = 4\pi C^2/(K-1)^2,$$

and assuming also a Hall effect $(\alpha_1, \alpha_2, \alpha_3)$ of type given by

$$u' = \sigma P - \alpha_3 Q + \alpha_2 R$$

in the current of conduction (u', v', w'), we have for the relation

between the total current (u, v, w) and the electric force (P, Q, R) the scheme

$$u = \frac{df'}{dt} + (4\pi C^2)^{-1}\frac{dP}{dt} + u'$$
$$= (4\pi C^2)^{-1}\left(K'\frac{dP}{dt} - a_3\frac{d^2Q}{dt^2} + a_2\frac{d^2R}{dt^2}\right) - \alpha_3 Q + \alpha_2 R;$$

the substitution of this relation, peculiar to the medium, in the fundamental circuital electrodynamic relations leads to the equations of propagation. It is to be noticed that for waves of high period (much higher however than ordinary light*) the constant corresponding to the Hall effect becomes inoperative. As already remarked, this scheme is directly applicable to the solution of the problem of reflexion without any complication arising with respect to interfacial conditions. The analytical scheme of equations adopted by Drude§ as a basis can be transformed into this shape, his rotatory coefficient (b_1, b_2, b_3) becoming equal to $K'^{-1}(a_1, a_2, a_3)$ and $(\alpha_1, \alpha_2, \alpha_3)$ being absent. It appears also that the present scheme is effectively the same as an earlier one adopted by Goldhammer, provided his rotatory coefficient (μ_1, μ_2, μ_3) is given by $\mu_1 K' d^2/dt^2 = a_1 d^2/dt^2 + 4\pi C^2 \alpha_1$; so that priority in formulating an adequate system of equations, which can satisfy the interfacial conditions by means of the ordinary electric variables without the intervention of an electromotive pressure, rests with him. The examination which is essential to make certain that the rotational term in the relation connecting polarization with electric force shall not involve perpetual motions, as would be the case with an ordinary statical rotational term, was made by Willard Gibbs as long ago as 1883†: he was led to retain the possibility of such a rotational scheme in the course of a very general discussion of the formal character of the reaction of the matter on radiation, but did not carry it into any detail; the circumstance noticed by him, that a medium constituted in that manner would also transmit waves of other than the optical type, was

* Cf. Leathem, *Phil. Trans.* 1897 A, for a full discussion of the subject.

§ Cf. British Association Report, 1893, *loc. cit.* §§ 15, 20.

† Cf. *loc. cit.*, § 16.

an indication that the problem of optical reflexion would be liable to complication by difficulties of the kind above mentioned.

133. The nature of structural optical rotation will now be very briefly considered. The rotational terms in the energy of polarization must be, in an isotropic medium, of the form‡

$$C\left\{f'\left(\frac{dh'}{dy}-\frac{dg'}{dz}\right)+g'\left(\frac{df'}{dz}-\frac{dh'}{dx}\right)+h'\left(\frac{dg'}{dx}-\frac{df'}{dy}\right)\right\}.$$

The relation between polarization and electric force is therefore by an argument similar to that of § 127, of type†

$$f'=\frac{K-1}{4\pi C^2}P+2C'\left(\frac{dQ}{dz}-\frac{dR}{dy}\right). \qquad C'=C\left(\frac{K-1}{4\pi c^2}\right)^2$$

The velocities of the two kinds of circularly polarized waves are now found to be

$$c\left(K\pm \overset{8\pi c^2 C'}{\cancel{C}}pK^{\frac{1}{2}}c^{-1}\right)^{-\frac{1}{2}}; \qquad \text{cf. App. F, § 5.}$$

and the rotational power of the medium as ordinarily measured is proportional simply to C. If we assume that the rotation in fluids is proportional to the number of active molecules, the result will be that for pure substances the rotation per unit length is jointly proportional to $(\mu^2+2)^2$ and to the density of the active substance*. The actual value of this coefficient is however found to vary very widely with temperature. The only

‡ Cf. *Phil. Trans.* 1894 A, p. 745.

† For a crystalline medium, by taking the most general possible quadratic function for the energy, it may be shown similarly that the rotational part of (f', g', h') is of form $\left(b\frac{dR}{dy'}-c\frac{dQ}{dz'},\ c\frac{dP}{dz'}-a\frac{dR}{dx'},\ a\frac{dQ}{dx'}-b\frac{dP}{dy'}\right)$, where $\frac{d}{dx'}, \frac{d}{dy'}, \frac{d}{dz'}$ represent arbitrary linear functions of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, and a, b, c are constants. In symmetrical crystals the forms are of course further restricted.

It is to be noticed that in an isotropic medium there can be no rotational effect in *statical* cases, because the electric force will then have a potential: it is only for waves of high frequency that it can arise.

* For solution in a neutral solvent, so dilute that μ is practically constant, the rotation per molecule would of course be constant: while for different solvents it should vary as μ^2+2. [According to the experiments of Pottevin, *Journ. de Phys.* July 1899, the rotation actually changes when solvents are mixed, in a manner which depends on their relative proportions, but is more complex than this law would imply.]

fundamental relation yet announced seems to be Gernez's† result that on plotting the curve of variation of the rotation per molecule with temperature, no discontinuity enters on passing from the liquid to the gaseous state, contrary to the above which would make the molecular rotation proportional to $(\mu^2 + 2)^2$: this would imply that C is independent of μ. It also appears that in the case of newly formed substances the rotation goes on sensibly altering for a considerable time.

The rotatory influence on the reflexion of radiation from the surface of a chiral medium is in actual cases inappreciably small: on attempting to deduce an expression for it we should find that, in problems such as this, in which spacial differentiations of orders higher than the second occur in the dynamical equations, the transition between two media cannot be mathematically treated as an abrupt interface subject only to displacement and traction.

134. In crystals the optical chirality is sometimes inherent in the arrangement of the molecules, being destroyed on fusion. It appears however that an inference would not be warranted that in such a case it is the crystal only, and not the individual molecule as well, that is in any way chiral, for absolutely non-chiral molecules could hardly spontaneously form a chiral structure. Thus there can be chirality of configuration in a molecule which does not involve chirality in its vibratory interaction with radiation.

The origin of the intrinsic rotational property in organic compounds has been identified with the presence, in the chemical space-formula of the molecule, of a carbon tetrahedron which has different elements at its four corners and has thus a configuration chirally different from that of its optical image. It would seem that to render this explanation complete, for optical rotation as distinct from statical crystalline plagiedry, we must make a call on the orbital or cyclic motions in the molecule**: for as regards simple vibrational properties a

† *Annales de l'Ecole Normale*, Vol. ii: confirmed by Guye and Aston, *Comptes Rendus*, Nov. 1897.

** It was noticed long ago by Lord Kelvin that the linear equations of an ordinary elastic medium cannot include rotational property.

static system without steady cyclic momenta is equivalent to its optical image, neither the potential nor the kinetic energy being affected by change of direction of *all three*‡ coordinate axes so long as the kinetic energy of vibration is a quadratic function of the velocities, the potential energy being always a quadratic function of the displacements. Such systems, thus optically non-rotational, may be chirally distinguishable by their geometrical form, which is determined by the complete molecular forces of which only the unbalanced parts affect the vibrations of the molecule. In the process of chemical synthesis from non-rotational elements, as many right-handed as left-handed molecules ought nevertheless in both cases (after Pasteur) to appear, unless the reagents are themselves rotational; and there would be no way of separating them except by rotational reactions or by crystallization.

A fundamental case of chirality occurs in electrolysis, an atom when a positive ion being the reflected image of the same atom when a negative ion. Generally, the change from a ~~molecule to its~~ enantiograph involves not merely perversion of its orbital configuration but also change of sign of each of its electrons: for, independently of any special aether-theory, the structure of an electric charge is indicated by the effects of disturbing it, which are chirally opposite for positive and negative charges.

The fact that the optical power of newly formed liquids continues to alter gradually for a long time after the reaction is complete is evidence that the rotational unit is not always a single molecule but may be a more or less loosely associated molecular complex: this would explain the absence of any definite general relation, in the case of a pure substance, between rotational power and temperature, as with rise of temperature the complexes would naturally be partly decomposed: were it not for Gernez's definite observation (*supra*) it might also be held to explain the absence of any observed connexion

‡ A property may be apparently affected by one perversion, yet if it is not altered by three successive perversions it will not be rotational: for three or any other odd number of them are equivalent to a single one together with a change of position in space.

between the rotational coefficient for a pure substance and its density.

The fact above demonstrated (§ 126) that the natural rotational coefficient both in crystals and in fluids can only depend on the space-gradient of the electric force, and not on that force itself, seems also to strengthen the presumption that it is often molecular aggregates that are involved in the effect, as is definitely known to be the case for those crystals which lose the property on fusion. If the property resided in the individual molecule, and each molecule had one chiral axis, the rotational power of the fused substance should be one third* of the maximum value for the crystal: while if the molecule were equally chiral about all axes, like a regular tetrahedral crystalline form with similar oblique facets on all the corners, the two should be equal.

As regards both kinds of rotation, it is to be remarked that the exciting cause is excessively minute compared with the causes of other optical phenomena. It is therefore not surprising that chemical isomers exist which differ only in rotatory power and are indistinguishable in their other physical qualities: the fact that the rotational coefficients are equal and opposite is often the only evidence that such isomers are exact enantiomorphs. The smallness of the optical effect in comparison with the obvious difference in crystalline form will not be remarkable on the present view, according to which there is no direct connexion between them.

135. As regards general characters, the molecules or molecular groups that show intrinsic rotatory power are necessarily included in the family which differ from their images by reflexion. Now it has been seen that, whatever theory of electricity may be adopted, the enantiomorph of a positive charge is an equal negative one: it appears then that, on the kinetic idea of a molecule, enantiomorphy reverses the signs of all its electrons and perverts their relative positions, while retaining their orbital characteristics. This consideration

* This assumes that the undirected axial rotational quality in the molecule is resolved by multiplication by the square of the cosine.

tends to confirmation of a dynamical explanation that would connect chirality with difference of directions in which the orbits of the electrons are described with respect to the mean configuration of the molecules.

The experiments of Bose (*Roy. Soc. Proc.* 1897), on the chirality of spirally twisted fibrous substances with regard to short Hertzian waves, are on a different footing. The chiral element is here a statical structure of the same order of dimensions as a wave-length, instead of a kinetic molecule so small in comparison with the wave-length as to act only by sympathetic vibration. It is rather an analogue of Reusch's artificial chiral optical system, built up of non-chiral crystalline plates arranged in spiral fashion.

According to the view here suggested, optical chirality such as exists in quartz could not be introduced by merely statically chiral molecular structure: for it has been seen (p. 206, footnote) that such structure cannot give rise to the rotational optical term. The examples just quoted, in which it is effective, are cases in which the degree of coarseness of grain of the medium is of the order of the wave-length of the radiation that is affected.

CHAPTER XIII

INFLUENCE OF THE EARTH'S MOTION ON ROTATIONAL OPTICAL PHENOMENA

136. In an examination of the influence of the Earth's motion on the rotation of the plane of polarization of light by a transparent substance, it is convenient to include both structural and magnetic rotation in the same analysis. It will suffice for practical purposes, and avoid much complication, to restrict the analysis to the simple case of plane waves travelling in the direction of the velocity v of translation of the material medium through the aether, and to take the rotational axis of the active medium in the same direction. If this direction is chosen as the axis of x we shall have f, f', P, a, and α null, as also all differential coefficients except those with respect to x. The relation between electric polarization and electric force will thus be

$$g' = \frac{K-1}{4\pi C^2} Q + \frac{\epsilon}{4\pi C^2} R,$$

$$h' = \frac{K-1}{4\pi C^2} R - \frac{\epsilon}{4\pi C^2} Q,$$

ϵ being a differential operator of the form $\epsilon_1 \frac{\delta}{dt} + \epsilon_2 \frac{d}{dx}$ in which ϵ_2 represents structural rotational quality and ϵ_1 magnetic rotation. As it is the time-rate of change of the state of the convected material medium, not of the stagnant aether, that determines the magnetic rotation, it is δ/dt, that is

$d/dt + v d/dx$, that is involved along with ϵ_1 when the coordinate axes are fixed in the aether. It is likely that other rotational terms will also occur, involving higher odd powers and products of the differential operators: but it will appear that those here given are sufficient for the purpose of the present argument, the inclusion of higher terms being simply equivalent to making ϵ_1 and ϵ_2 themselves functions of the period and wave-length.

137. The dynamical equations of aethereal propagation are the circuital relations, which by our general theory of moving media (§ 73) assume the form, with reference to axes fixed in the aether,

$$-\frac{dR}{dx} = -\frac{\delta b}{dt}, \quad \frac{dQ}{dx} = -\frac{\delta c}{dt}$$

and

$$-\frac{d\gamma}{dx} = 4\pi v = 4\pi\left(\frac{dg'}{dt} + \frac{dg}{dt}\right)$$

$$\frac{d\beta}{dx} = 4\pi w = 4\pi\left(\frac{dh'}{dt} + \frac{dh}{dt}\right);$$

wherein

$$4\pi C^2 g = Q + vc,$$

$$4\pi C^2 h = R - vb$$

$$4\pi C^2 g' = (K-1)Q + \epsilon R$$

$$4\pi C^2 h' = (K-1)R - \epsilon Q$$

and, as the electrodynamic effect of convection is treated as magnetism,

$$\beta = \frac{b}{\mu} - 4\pi v h',$$

$$\gamma = \frac{c}{\mu} + 4\pi v g',$$

the magnetic constant μ being here retained** in view of possible applications in the domain of long Hertzian waves.

** It is assumed here that the relation of the induced magnetization to (a, b, c) is not modified on account of the convected polarization: in the practical case of non-magnetic media no assumption is however involved. When there is such *quasi*-magnetism present, it seems natural to consider the induced magnetism as directly conditioned by the physical vector (a, b, c): it is certainly not then a function of the (α, β, γ) defined in § 73 alone. It will appear (§ 140) that this view fits in with the general theory.

The second of these systems of circuital equations thus gives

$$-\left(\frac{C^2}{\mu}\frac{d}{dx}+v\frac{d}{dt}\right)c = K\frac{\delta Q}{dt} - v\frac{dQ}{dx} + \frac{\delta}{dt}\epsilon R$$

$$\left(\frac{C^2}{\mu}\frac{d}{dx}+v\frac{d}{dt}\right)b = K\frac{\delta R}{dt} - v\frac{dR}{dx} - \frac{\delta}{dt}\epsilon Q.$$

Substituting for b, c from the first circuital relation, the equations of propagation of electric force (that is, of electric material polarization) are obtained. If we write Θ for the complex variable $Q + \iota R$, they combine into the single equation

$$\frac{d}{dx}\left(C^2\frac{d}{dx}+\mu u\frac{d}{dt}\right)\Theta = \left(K\mu\frac{\delta^2}{dt^2} - \mu v\frac{d}{dx}\frac{\delta}{dt} - \iota\mu\epsilon\frac{\delta^2}{dt^2}\right)\Theta.$$

For a train of circularly polarized waves, Θ or $Q + \iota R$ is proportional to $\exp \pm \iota\frac{2\pi}{\lambda}(x - Vt)$, the positive sign representing right-handed rotation of the electric force as the wave advances along the axis of x, and the negative sign left-handed rotation. The substitution of this form leads, after transposition, to the equation

$$C^2 - \mu v^2 = K\mu(V-v)^2 + 2\mu v(V-v)$$
$$\mp\frac{2\pi\mu}{\lambda}(V-v)^2(\epsilon_1 V - \epsilon_1 v - \epsilon_2),$$

which determines the velocity V of propagation of the waves, the result differing according as their polarization is right-handed or left-handed: in obtaining this equation no approximation has been employed.

138. If there is no rotational quality at all, the equation becomes

$$V^2 - 2(1-K^{-1})vV + (1-K^{-1})v^2 = V_0^2,$$

where V_0^2 or $C^2/K\mu$ is the usual velocity in the medium; leading to

$$V = V_0 + (1-K^{-1})v - (K^{-1}-K^{-2})v^2/2V_0$$

approximately, as in § 36; of which the second term is Fresnel's expression for the effect of convection, and the third is a second-order term which will combine with other effects of that order.

139. If there is rotation of purely structural type ϵ_1 is null: the effect of ϵ_2 as regards velocity of propagation is, by the equation for the velocity, exactly the same as that of increasing K by $\pm 2\pi\epsilon_2/\lambda$, the upper sign applying to a right-handed circular wave, the lower to a left-handed one, the one case differing from the other only in the sign of the rotatory coefficient ϵ_2. This modification of K does not involve the velocity v of translation of the medium at all, and continues to be a method of representing the rotatory effect of the medium when there is no material convection: the additional effect arising from the motion of the rotational medium is therefore to modify the velocity of each of the existing circular wave-trains exactly as if it were light travelling in an ordinary medium, that is, in accordance with Fresnel's law.

Thus the velocity V_1' *relative to the moving material medium*, of a right-handed wave, of length λ referred to the resting aether, is given, to the first order, by

$$V_1' = \frac{C}{\mu^{\frac{1}{2}} K_1^{\frac{1}{2}}} - \frac{v}{K_1},$$

where

$$K_1 = K + \frac{2\pi\epsilon_2}{\lambda};$$

so that, up to the order $\epsilon v/C$ but not including $(v/C)^2$,

$$\begin{aligned} V_1' &= \frac{C}{(\mu K)^{\frac{1}{2}}}\left\{1 - \frac{\pi\epsilon_2}{\lambda} + \frac{3}{2}\left(\frac{\pi\epsilon_2}{K\lambda}\right)^2\right\} - \frac{v}{K}\left(1 - \frac{2\pi\epsilon_2}{K\lambda}\right) \\ &= V'\left\{1 - \left(1 - \frac{v}{KV'}\right)\frac{\pi\epsilon_2}{K\lambda} + \frac{3}{2}\left(\frac{\pi\epsilon_2}{K\lambda}\right)^2\right\}, \end{aligned}$$

where

$$V' = V_0 - \frac{v}{K}.$$

The effect of the convection, up to the first order, is thus to reduce the rotatory coefficient in the ratio $1 - v/KV'$, while it at the same time alters the mean velocity relative to the medium from V_0 to V', equal to $V_0 - v/K$, in accordance with Fresnel's principle. This change in the value of the rotatory coefficient agrees with the result of an investigation by Lorentz*: but, overlooking the effect of the concomitant change

* 'Versuch...,' pp. 118—119: the difference in sign is probably only apparent.

of velocity, he inferred that the convection would produce a first-order effect on the actual structural rotation, in contradiction to the experimental result of Mascart (§ 142).

140. To examine this point, it will be most convenient to make direct use of the result just proved, that the convection of the rotational medium merely modifies the velocities of each of the circular waves of permanent type in the ordinary manner. Let U_1 and U_2 be the velocities of right-handed and left-handed circular waves in the actual material medium when at rest; then their velocities *relative to it* when it is in uniform motion with velocity v in the direction of the ray are V_1' and V_2' where, retaining second order terms for the sake of possible future applications,

$$V_1' = U_1 - m_1^{-2}v - (m_1^{-2} - m_1^{-4})\, v^2/2U_1,$$
$$V_2' = U_2 - m_2^{-2}v - (m_2^{-2} - m_2^{-4})\, v^2/2U_2,$$

m_1 and m_2 being the respective refractive indices when the medium is at rest, so that $m_1U_1 = m_2U_2$. This follows by Fresnel's formula of § 36; or from the result of § 137 by substituting m^2 for $K \pm 2\pi\epsilon_3/\lambda$ and U_1, U_2 for c/m_1, c/m_2, and following the procedure of § 138. If λ_1 and λ_2 are the respective wave-lengths when the medium is at rest (so that, τ being the period, $\lambda_1 = U_1\tau$, $\lambda_2 = U_2\tau$), the wave-lengths in the moving medium relative to that medium are λ_1' and λ_2' where

$$\lambda_1' = \frac{V_1'}{U_1}\lambda_1 = \lambda_1\left\{1 - m_1^{-2}\frac{v}{U_1} - (m_1^{-2} - m_1^{-4})\frac{v^2}{2U_1^2}\right\},$$

with the similar expression for λ_2', the period of the light relative to the moving system being unaltered by the motion when the radiating source is terrestrial and thus partakes in the motion. In passage across a length l of the rotational medium, the one wave has gained on the other by N' periods of either, where

$$N' = l/\lambda_1' - l/\lambda_2'$$
$$= \frac{l}{\lambda_1} - \frac{l}{\lambda_2} + lv\left(\frac{1}{\lambda_1 m_1^2 U_1} - \frac{1}{\lambda_2 m_2^2 U_2}\right) + \tfrac{1}{2}lv^2\left(\frac{1}{\lambda_1 m_1^2 U_1^2} - \frac{1}{\lambda_2 m_2^2 U_2}\right)$$
$$+ \tfrac{1}{2}lv^2\left(\frac{1}{\lambda_1 m_1^4 U_1^2} - \frac{1}{\lambda_2 m_2^4 U_2^2}\right),$$

correct to the second order of small quantities.

Since $\lambda_1/\lambda_2 = U_1/U_2 = m_1^{-1}/m_2^{-1}$, this becomes

$$N' = N\left\{1 + \tfrac{1}{2}(m^{-2} + m^{-4})\frac{v^2}{U^2}\right\},$$

where N is the corresponding gain in periods when the medium is at rest, it being now unnecessary to retain the subscripts in m and U. Thus up to the first order of v/U the optical rotation is unaltered by the motion of the medium, as Mascart found it to be*, and as our present general theory (§ 112) requires: so that there is no discrepancy as regards these rotational phenomena. The second-order proportional alteration here obtained, arising from the effect of the convection of the medium on the waves, depends only on the refractive index: but, as in all such cases, it will have to be combined with an unknown alteration of the same order arising from the intrinsic change in the rotational coefficient of the medium which is produced by its motion through the aether. On the special electron theory, the analysis of § 112 would make out that these alterations, combined with the intrinsic alteration of material dimensions arising from the motion of the medium, should exactly compensate so as to give a null result.

141. The effect of rotation of magnetic type on the velocity of a circularly polarized wave is obtained by making ϵ_2 null in the general formula. The result cannot be expressed so simply as in the previous case; and on account of the actual smallness of the magnetic type of rotation there is not much object in pursuing it in detail up to a second approximation. To that approximation the value of V given by the general equation comes out as

$$V = V_0 + (1 - K^{-1})v \pm \left(1 - \frac{3v}{KV_0}\right)\frac{\pi V_0^3}{K\lambda}\epsilon + \frac{5}{2}\frac{\pi^2 V_0^3}{K^2\lambda^2}\epsilon^2 - (K^{-1} - K^{-2})\frac{v^2}{2V_0}.$$

For the velocity V' of waves of length λ' relative to the

* The fact that Mascart's null result would be accounted for by assuming that Fresnel's law applied exactly to each circular component of the light was demonstrated by Ketteler, 'Astronomische Undulationstheorie' 1873, p. 106.

medium, we have $V' = V - v$, where V is given by the above expression in which K is equal to m^2, and λ' is written in place of λ. When the medium is at rest, let the corresponding velocity be U; the length of the wave will then be λ, equal to $U\lambda'/V'$; and U is the value of V' when in it v is made null and λ is written for λ'. Thus

$$\lambda\epsilon = \lambda'\epsilon U/V'$$
$$= \lambda'\epsilon(1 + v/m^2V_0) \text{ to the second order};$$

therefore $$U = V_0 \pm \left(1 - \frac{v}{m^2V_0}\right)\frac{\pi V_0^2\epsilon}{m^2\lambda'} + \frac{5}{2}\frac{\pi^2V_0^3\epsilon^2}{m^4\lambda'^2};$$

so that

$$V' = U - m^{-2}v \mp \frac{2\pi V_0}{m^4\lambda'}v\epsilon - (m^{-2} - m^{-4})\frac{v^2}{2V_0}.$$

In this formula m, or $K^{\frac{1}{2}}$, is the refractive index when the velocity is V_0 or $c/K^{\frac{1}{2}}$: we must express it in terms of m_1, the index when the velocity is the one, U, belonging to the actual type of circular wave when the medium is at rest so that $m_1U = mV_0$; this gives

$$m^{-2} = m_1^{-2}(1 \mp 2\pi V_0\epsilon/m_1^2\lambda')$$

to the first order. Hence finally

$$V' = U - m_1^{-2}v - (m_1^{-2} - m_1^{-4})\frac{v^2}{2V_0}.$$

Thus, in the case of magnetic rotation also, the effect of the motion of the material medium on the rotational phenomena relative to the moving medium can be correctly obtained, up to the order which includes the product $v\epsilon$, by applying the Fresnel correction to the velocity of each circularly polarized wave separately. The argument of § 140 then shows that in this case also the value of the rotational coefficient is unaltered up to this order $\epsilon v/c$ by the convection of the material system.

142. In the investigation for quartz above referred to, Lorentz has contemplated the formal possibility of the interaction between the rotational structure of a naturally active medium and the directed quality of its convection introducing

a small rotation of the magnetic type*. The negative result of Mascart, in conjunction with the present analysis, leads to the conclusion that this does not really exist.

The experiments of Mascart† in fact revealed no influence of the Earth's motion on the optical rotation produced by quartz, with radiation from a given terrestrial source, although an alteration of 5×10^{-5} of the actual rotation would have been detected. If there were really a first-order influence of the Earth's motion on the value of ϵ, it must be of the order v/V or 10×10^{-5} of the total amount. Thus though there is not a great deal to spare in precision, the experiments materially support our theoretical conclusion that there is no first-order effect.

The first-order effect on naturally rotational media that has here been proved non-existent, is one involving the product of ϵ and v/V. If there were an *à priori* reason assignable to show that the alteration produced in the rotation is unaffected by reversal of the velocity of convection of the matter, we might have inferred the absence of this term at once. If however we allowed an analogy between the advance of a structurally rotational material system through the aether and the advance of a spiral screw through a cork, we should on the contrary anticipate the presence of such a term; this indicates that limitations must beset the employment of static or geometrical spiral models (cf. § 134) in the kinetic departments of stereochemical theory.

143. The general analysis of Ch. XI, resting on the basis that the motion of electrons is the cause of all electrodynamic and optical phenomena, led to the conclusion that the structure of molecules and the form of material bodies is not altered by motion through the aether, up to the first order in v/C, and thence to the result that no optical observations which depend on making an adjustment to cut off the light can be affected up to this order in v/C. This general theory involves the

* This is in fact the first-order influence of which the possibility was suggested in the discussion on symmetry, § 92.

† *Annales de l'École Normale*, 1872.

negative result of Mascart's experiments**. As previously pointed out (§ 92), there was *à priori* no geometrical or formal consideration, as distinct from dynamical, that would exclude an intrinsic proportional alteration in the rotatory coefficient, of the order of $\epsilon v/C$ but of the magnetic type, arising from the convection of the material medium, in addition to the direct reaction with the radiation that has here been found to be null. The experimental evidence, however, independently of this analysis, pointed to the conclusion that alteration of ϵ, of either type, is non-existent, because the two types could hardly be expected to compensate each other exactly over a series of observations taken at different times.

The null result of Mascart is thus important in connexion with this view that electrodynamic disturbances arise wholly from the motions of electrons, which in fact requires its validity. In Lorentz's treatment of the correlation between a uniformly moving electrodynamic system and a stationary one, the foundation was apparently considered to be not sufficiently wide to cover the case of systems involving rotational property; while a direct investigation seemed to indicate a discrepancy in that case. In the present procedure the argument is wholly based on the electron theory, and if its result had not been experimentally confirmed, it would have been a question whether the electric structure of rotational media was consistent with their being constituted solely of electrons: as things are, the agreement with fact, which there was apparently nothing in the shape of general argument to foreshadow, carries with it an independent presumption of the effective validity of that view. And if this presumption were not admitted to be a substantial one by itself, it would derive cumulative value when put in connexion with the fact that the ascertained Amperean mechanical forces between linear currents have been theoretically established only on the hypo-

** It also includes the null result of § 141 for magnetic rotation, notwithstanding that by § 112 convection modifies the magnetic field by adding a term involving (f, g, h); for here the imposed magnetic field which induces the rotation is supposed so great that the magnetic field of the radiation need not be added to it in forming the rotational term, so that a modification in it depending on the (f, g, h) of the radiation is also negligible.

thesis that currents of conduction are formed by the convection of discrete electric particles*. With a view to strengthening this position further, it would perhaps be desirable if possible to push the precision of Mascart's null result up to the next power of ten, so as to make its application quite certain.

The theory of the null effect of the Michelson interference experiment is not so certain as those above considered, for it carries us into the region of the second order of v/C, and the changes of dimensions and physical constants of material systems which probably exist to that order. Yet the two effects lend each other a certain amount of mutual support. The experiment of Fizeau in which he obtained evidence of a change in the deviation of the plane of polarization after transmission through a pile of glass plates, concomitant with a change in the direction of the beam with respect to the Earth's motion, is of a different order of importance, in that the null effect anticipated by theory was there simply not attained owing presumably to the acknowledged difficulty of eliminating disturbing causes. Thus no relation has yet presented itself which is not consonant with the present general theory, involving mobile electrons in a stagnant aether; while on the other hand there is no competing theory that is at all complete or coherent.

* *Phil. Trans.* 1895 A, p. 698; cf. also Appendix E.

SECTION V

CHAPTER XIV

ON THE MECHANISM OF MOLECULAR RADIATION

144. THE theory of electrical actions which ascribes the electrodynamic effects of electric currents solely to the motion of electrons is the only dynamical theory yet suggested whose consequences are not in discrepancy with the facts relating to electrodynamic attractions**. According to that view, every disturbance of the aether, including radiation as one type of disturbance, is originated by translatory motion of electrons through the aether. This puts us in a position to attempt a theory of the mechanism of radiation, if we first obtain complete expressions for the aethereal disturbance initiated by and propagated from a single moving electron.

Except at places whose distance from the nucleus of the electron is so small as to be comparable with the linear dimensions of the nucleus itself, it is sufficient to consider the electron as a point-charge; and the aethereal disturbance arising from the motion of the electron is to be obtained by simple superposition of elementary disturbances arising from its transit over the successive elements of its path†. Suppose then that an electron e is at the point A and after a time δt is at B, where $AB = v\delta t$, v being its velocity; the effect of its

** Cf. *Phil. Trans.* 1895 A, p. 698. More recently, direct experimental evidence in favour of such a theory has been accumulating.

† What follows is mainly adapted from a paper 'On the Magnetic Influence on Spectra; and on the Radiation from moving Ions,' *Phil. Mag.* Dec. 1897.

change of position is the same as that of the creation of an electric doublet AB of moment $ev\delta t$; thus we have only to find the disturbance introduced by the creation of such a doublet, and then integrate the result along the paths of all the electrons of the vibrating molecule.

145. Consider therefore such a doublet at the origin, lying along the axis of z; for it, or indeed for any distribution symmetrical with respect to that axis, the lines of magnetic force will be circles round the axis, and the force will be specified by a single variable, its intensity H. The electric current, whether in dielectric or in conducting media, will circulate in wedge-shaped sheets with their edges on the axis, and may be specified by a stream function, as in fact will directly appear. If we employ cylindrical coordinates ρ, ϕ, z, and apply the Amperean circuital relation (viz. circulation of magnetic force equals 4π times current) to the faces of the element of volume $\delta\rho \,.\, \rho\delta\phi \,.\, \delta z$, we obtain for the components P, R of the electric force

$$\frac{d\mathrm{P}}{dt} = -C^2 \frac{dH\rho}{\rho dz}, \quad \frac{dR}{dt} = C^2 \frac{dH\rho}{\rho d\rho},$$

so that $H\rho$ plays the part of a stream function; while by the circuital relation of Faraday we have also

$$\frac{d\mathrm{P}}{dz} - \frac{dR}{d\rho} = -\frac{dH}{dt}.$$

Thus the characteristic equation for H is

$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho H + \frac{d^2H}{dz^2} = \check{C}^2 \frac{d^2H}{dt^2},$$

which is

$$(\nabla^2 - \rho^{-2})\, H = \check{C}^2 \frac{d^2H}{dt^2},$$

where ∇^2 is Laplace's operator. But a more convenient reduction comes on substituting $H = dY/d\rho$, and then neglecting an irrelevant operator $d/d\rho$ along the equation: this gives

$$\nabla^2 Y = \check{C}^2 d^2 Y/dt^2.$$

146. We can now express in a general manner the disturbance emitted by an electric doublet situated along the axis of z at the origin, and vibrating so that its moment M is an arbitrary function of the time. As regards points near it, at which the field is immediately established, the doublet may be treated as a linear current-element of strength dM/dt: close up to such an element, in its equatoreal plane, the magnetic force H due to it is $-r^{-2}dM/dt$. The appropriate solution for Y to fit this simplest case is

$$Y = r^{-1}f(t - r/c),$$

so that

$$H = -\sin\theta\left\{\frac{f(t-r/c)}{r^2} + \frac{f'(t-r/c)}{cr}\right\},$$

giving when θ is $\frac{1}{2}\pi$ and r is very small, $H = -r^{-2}f(t)$; thus $dM/dt = f(t)$. That is, if the mode of change of the moment of the oscillating doublet is given in the form $dM/dt = f(t)$, the magnetic force thus originated at the point (r, θ) is

$$H = -\sin\theta\left\{\frac{f(t-r/c)}{r^2} + \frac{f'(t-r/c)}{cr}\right\}$$

$$= \sin\theta\,\frac{d}{dr}\frac{1}{r}f\left(t - \frac{r}{c}\right).$$

The second term in H is negligible near the origin for movements not excessively sudden, as it involves the velocity c of radiation in the denominator: but at a sufficiently great distance it is the chief term.

The components of the magnetic field due to a vibrating doublet M at the origin, whose direction vector is (l, m, n), are therefore, at points close to the doublet,

$$(mz - ny,\, nx - lz,\, ly - mx)\,\frac{1}{r}\frac{d}{dr}\frac{1}{r}f\left(t - \frac{r}{c}\right),$$

where
$$\frac{dM}{dt} = f(t);$$

and the components of the magnetic field, that is of the disturbance, emanating from any system of electric oscillators vibrating in any given manner, can thence be expressed in a

general formula of integration. At present we only want the effect of sudden establishment of the doublet $M, = ev\delta t$, at the origin. This comes by integration over the very small time of establishment; there is a thin spherical shell of magnetic force propagated out with velocity c, the total force in it when integrated across the shell being *exactly* $-Mr^{-2}\sin\theta$ at all distances, whatever be the thickness of the shell which will of course depend on the actual time taken in the establishment of the doublet; this follows because the integral of the second term in H vanishes, dM/dt being null at the beginning and the end of the operation. The aggregate amount of magnetic force propagated in the spherical sheet is thus the same as that of the steady magnetic force for the time δt due to a permanent steady current-element of intensity $M/\delta t$ or ev: it is clear, in fact, that this must be so, if we consider a sudden beginning of this permanent current-element and remember that its magnetic field establishes itself by spreading out from it ready formed with the velocity of radiation.

147. The magnetic force at a point at distance r due to a moving ion thus depends on the state of the ion at a time r/c previously; for *near* points it is in the plane perpendicular to r, at right angles to the projection v of the velocity v of the ion on that plane, and equal to ev/r^2. For vibrations whose wave-length in free aether is very great compared with the dimensions of the ionic orbit in the molecule, if we interpret magnetic force as velocity of the aether, the vibration-path of a point attached to the aether, and close to the vibrator, will be in the plane transverse to the radius vector r, and will be similar to the projection of the orbit of the electron on that plane when turned round through a right angle.

148. Further away from the ion the law of variation of the magnetic force with distance is $ev/r^2 + e\dot{v}/cr$ instead of ev/r^2. Thus at a distance of a large number of wave-lengths, the vibration-curve of the radiation proper, which as we have seen is constituted of an alternating shell of radiation for each single impulse $ev\delta t$, is similar to the projection of the hodograph of

the orbit of the ion on the wave-front, instead of the projection of the orbit itself.

It thus appears that when the orbital motions in a molecule are so constituted that the vector sum $|\Sigma e\dot{v}|$ of the accelerations of all its electrons, with due regard to their signs, is constantly null, there will be no radiation, or very little, abstracted from it, and therefore this steady motion will be permanent. The condition that is thus necessary for absence of dissipation by radiation limits the number of types of motions otherwise steady in the molecule, that can be permanent: for example, in the orbital motion of two electrons of equal inertia and opposite charge, round each other, the accelerations reinforce each other instead of cancelling, so that this simple type is not a possible permanent molecular conformation, though it is easy to construct other steady types that would be possible.

As the vibration of a near point in the aether is thus similar to the projection on the wave-front of the resultant or aggregate of the vibrations of the electrons in the molecule, and the vibration at a distant point in the aether is similar to the projection on the wave-front of the aggregate of the motions of their hodographic points, it is verified that the intrinsic periods of the radiation are those of the system of ions that originate it.

If the condition that wave-length is very large compared with magnitude of the molecule were not satisfied, lag of phase would sensibly disturb these results so that the vibration though periodic would no longer be simple harmonic, and in effect each spectral line would be accompanied, more or less, by its system of harmonics.

149. This expression for the radiation from a system of moving electrons may be verified by applying it to the simple case treated by Hertz in his mathematical discussion of electric oscillators, namely that of a stationary rectilinear electric vibrator in which the electric moment oscillates harmonically between the values $+El$ and $-El$, with wave-length λ' and therefore period λ'/c: according to his result the radiation per half-period is $\pi^4E^2l^2/3\,(\frac{1}{2}\lambda')^3$*. To obtain the radiation per unit

* 'Electric Waves,' English edition, p. 150: his λ is $\frac{1}{2}\lambda'$.

time this must be divided by $\frac{1}{2}\lambda'/C$, yielding $16\pi^4E^2l^2C/3\lambda'^4$. If now we consider two electric charges $+e$ and $-e$ following each other round a circle of diameter l so as to be always at opposite points, they are equivalent to two such Hertzian oscillators in perpendicular planes. For each of these charges the time-rate of radiating energy (§ 150) is $\frac{2}{3}e^2C^{-1}\dot{v}^2$, where $\dot{v} = \frac{1}{2}l(2\pi C/\lambda)^2$: when we include both of them, since their vibrations are in the same phase the energy is quadrupled, instead of merely doubled as it would be if their phases varied arbitrarily from time to time; hence it is in all $32e^2l^2\pi^4C^3/3\lambda^4$, which agrees with the above result when it is remembered that e, being specified in electromagnetic units, is equal to E/C.

It is important to bear in mind that though the total energy emitted by a series of radiators with arbitrarily changing phases is on the average by Lord Rayleigh's theory the sum of the energies that would be due to them separately, yet this principle must not be applied to the electrons of a molecule whose phases are during each interval of undisturbed radiation in definite relations to each other.

150. Although the molecule as a whole is thus protected from loss of its energy by radiation when the vector sum of the accelerations of its electrons is permanently null, we have still to examine to what extent the motion of an isolated electron will have its energy drained away from this cause.

In consequence of the stream-function property of $H\rho$, the components of the time-gradient of the electric force, taken along δr and along $r\delta\theta$, are respectively

$$\frac{C^2}{\rho}\frac{dH\rho}{rd\theta} \text{ and } -\frac{C^2}{\rho}\frac{dH\rho}{dr},$$

ρ being $r \sin\theta$; thus they are

$$-2C^2\cos\theta\left\{\frac{f(t-r/C)}{r^3}+\frac{f'(t-r/C)}{Cr^2}\right\},$$

and $$-C^2\sin\theta\left\{\frac{f(t-r/C)}{r^3}+\frac{f'(t-r/C)}{Cr^2}+\frac{f''(t-r/C)}{C^2r}\right\};$$

and the value of the electric force is obtained by integrating these expressions with respect to t.

At a very great distance this electric force (as well as the magnetic force) is thus perpendicular to r, and is equal to $-r^{-1}\sin\theta f'(t-r/C)$; and the flow of energy arising from the disturbance is thus radial. For the case of an ion e moving with velocity v, $f(t)$ is equal to ev; and in $f(t-r/C)$ the value of the function f for any point belongs to the position of the source at a time r/C previous, where r is its distance at that time from the point at which the field is being specified. The rate of loss of energy by radiation may be computed by Poynting's formula as $(4\pi)^{-1}$ times the product of the above electric and magnetic forces integrated over an infinite sphere*: it is thus $(4\pi r^2 C)^{-1}\{f'(t-r/C)\}^2\int\sin^2\theta\, dS$, or $\frac{2}{3}e^2C^{-1}\dot{v}^2$, where $\dot{v}$ is the acceleration of the electron at a time r/C previously. This expression thus represents the amount of energy per unit time that travels away and is lost to the system, the velocity of the electron being as usual taken to be of a lower order than that of radiation.

In the process of setting up a velocity v of the electron from rest, there is thus a loss of energy by radiation, equal to $\frac{2}{3}e^2C^{-1}\int\dot{v}^2dt$. In motion with uniform velocity there is no loss; during uniformly accelerated motion the rate of loss is constant.

As the electric and magnetic forces at a great distance are each proportional to the acceleration of the electron at a time r/C previous, and do not involve its velocity, and as we can combine the components of its motion in fixed directions, it follows generally that for an isolated electron the rate of loss of energy by radiation is $\frac{2}{3}e^2C^{-1}$ multiplied by the square of its acceleration.

The store of kinetic energy belonging to the electron is of the order $\frac{4}{3}e^2a^{-1}v^2$ where a represents the linear dimensions of its nucleus, the expression being exact for a spherical nucleus. Thus the loss of energy by radiation from any steady orbital motion, in the interval between two disturbances (§ 161), would not in any case be sensible compared with its whole intrinsic

* The flow of radiation near the electron may be examined in detail, as Hertz has done in the similar problem of a linear vibrator. The two cases are equivalent *as regards kinetic effects*, for a positive electron oscillating around the origin becomes a vibrating doublet when an equal *stationary* negative electron is added at the origin. See Add. & Corr.

kinetic energy, when the velocities of the electrons are not of the order of magnitude of that of radiation: while for higher velocities the importance of the radiation is, in part at any rate, counteracted by the increase of the inertia coefficient which then occurs.

Absence of Radiation from undisturbed Molecules

151. The linearity of the aethereal equations allows of the superposition of solutions. Hence when an electron has been moving in any manner through the aether, the aethereal disturbance produced by it is made up by the superposition of spherical shells of magnetic force due as above to the different elements of the path of the moving nucleus. When its velocity is extremely great, of an order comparable with that of radiation, there will tend to be a crowding of these shells of magnetic force in front, and an opening up of them behind, which, involving a deviation from the distribution of magnetic and electric forces that is suitable for propagation onwards, will usually cause the throwing off of secondary waves backwards, so that each shell of disturbance will diffuse itself as it proceeds, instead of remaining of uniform thickness: this effect will however at any realizable speeds be of trifling amount and will finally in a steady state of motion disappear (§ 97).

When an electron is started from rest into motion these shells of magnetic force travel out from it in succession with the speed of radiation. After the motion has become of uniform rectilinear type, the energy in any one of the shells diminishes as it travels out, being always inversely as the square of its radius and so becoming negligible after a time. This can only arise, if we adhere for descriptive purposes to the notion of continuous transfer of energy, from the energy of the following shells being in part (in fact in this case wholly) recruited from those ahead instead of being entirely extracted from the moving nucleus: for it is to be remembered that quantities of energy, involving the squares of velocities, are not simply superposable as are the velocities themselves. It follows also that there is never any drag on the motion of a uniformly travelling electron on account of radiation.

If we had merely to do with the sudden creation of an electric doublet or a sudden small displacement of an electron, there would be only one isolated shell of magnetic force travelling out into the medium, and the energy in it would then of necessity remain constant. This may be verified as regards the kinetic part (and similarly as regards the potential part) if we integrate the square of the magnetic force H across the thickness of the shell. The complete value of H at the surface of a sphere of radius r, due to a moving electron that was at the centre of the sphere at a time r/C ago, is $-\sin\theta\,(ev/r^2 + e\dot{v}/Cr)$; of this the second term, which comes to nothing in the integration for H itself across the thickness of the shell of radiation, because it involves as much negative as positive, $\int \dot{v}dt$ being null, will be preponderant in the integral of H^2 across the shell when r is great, yielding the value $(Cr)^{-2}\int C\,(dM/dt)^2\,dt$ or $e^2/Cr^2\int \dot{v}^2dt$, which may be expressed as $e^2\tau/Cr^2$ multiplied by the mean square of the acceleration of the electron during its time of motion τ. Integrated over the whole spherical shell of radiation, the result, $\frac{2}{3}e^2/C\int \dot{v}^2dt$ as above, is independent of r so that the energy of the expanding shell is conserved as it moves.

Thus a single electron travelling without acceleration of its velocity does not radiate at all on account of its motion, and experiences no resistance: while sudden changes of velocity, for example the sudden stoppage of a rapidly moving electron, will originate shells of intense radiation.

152. To repeat in other words, when an electron is put into motion it sends out a stream of radiation which lasts as long as its velocity is being accelerated: when its velocity has become constant, there is no more radiant energy sent out from it, though the previous sheets of radiation will continue to travel on into the more distant stagnant aether, leaving behind them ready formed the steady magnetic field of the uniformly moving electron: but that field, which thus becomes established as a trail or residue of the shell of radiation arising from the original initiation of the motion of the electron, does not itself

involve any sensible amount of energy except in the immediate neighbourhood of the electron.** It is of importance to realize this process of formation of the magnetic field arising from a local electric disturbance, as it supplies the answer to an objection that has been offered to the treatment of a moving electron as possessing ordinary inertia, namely that it would require an infinite time after its motion has been started before all the surrounding aether could attain the steady state appertaining to that motion: the answer is that practically all the energy of the steady aethereal field thus belonging to the motion is in the immediate neighbourhood of the electron, where the field is established immediately, so that there is really no time-lag in the reaction of the aethereal disturbance on the electron, such as would arise if we had to await the adjustment of the distant parts of the field of disturbance after each change in its velocity.

153. The magnetic, or kinetic, part of the aethereal disturbance sent out by an electron moving in any manner thus consists of the following parts: a part depending on its velocity v, of which the element that is initiated in the time δt consists of a spherical shell of magnetic force travelling out with the velocity of radiation, whose aggregate intensity (magnetic force integrated across the shell) at any point is $-evr^{-2}\,\delta t \sin\theta$, where θ is the angle between the direction of v and that of the distance r of this point from the position the electron occupied at the time of emission of this shell of radiation: another part depending on its acceleration $\dot{v}$, given similarly by $-e\dot{v}\,(Cr)^{-1}\,\delta t \sin\phi$, where ϕ is the angle between $\dot{v}$ and r. As already mentioned, this result is subject to slight correction, initially of the order

** In the same way, when a moving electron is stopped, its magnetic field is wiped out by the shell of radiation which travels out from it owing to the retardation of its velocity. Even if it were stopped dead the kinetic energy of its field would not however all go off in radiation. On collision with a material wall it will displace the electrons of the wall, and their resilience may deflect it or it may be entangled among them; and much of the energy will be dissipated thermally in their irregular motions. But there can be no destruction of momentum on the impact.

v/C, but ultimately after a very minute time of the order $(v/C)^2$, arising from the crowding up of the shells of magnetic force in front of the moving electron; for that crowding may disturb the balance of the radiation so that some of it is thrown back again towards the source. To determine, to this degree of approximation, the aethereal disturbance arising from any system of electrons whose motion is given, is now merely a question of integration: but the movement of the sources of radiation themselves makes it a very complex one to handle except in special problems: general analytical formulae might be constructed, but it is one of those cases in which mathematical symbolism may darken counsel.

154. Let us now examine more minutely the aggregate contribution to the magnetic force at distance r, originated by the disturbances arising from the x-components of the motions of the various electrons in a molecule, each disturbance being of course sent out at a time r/C previous to the instant considered. We have, assuming rectilinear simple harmonic motion for simplicity,

$$v = A \cos 2\pi t/\tau, \quad \dot{v} = -2\pi A \tau^{-1} \sin 2\pi t/\tau:$$

thus the magnetic force at distance r arising from the electron e is H, equal to

$$-\frac{eA}{r^2} \sin (xr) \cos \frac{2\pi}{\tau}\left(t - \frac{r}{C}\right) + \frac{eA}{Cr}\frac{2\pi}{\tau} \sin (xr) \sin \frac{2\pi}{\tau}\left(t - \frac{r}{C}\right),$$

and at right angles to the plane xr. Suppose now we had a doublet oscillating along the axis of x, consisting of this electron and another one at distance a from it in the direction of that axis, for which the values of ev, and therefore of $e\dot{v}$, are at each instant equal and opposite. Their combined magnetic force is $-a dH/dx$: thus it involves two parts, one arising from differentiating the coefficient of the periodic term in H, the other from differentiating the periodic term itself. The former part has in its coefficient the same inverse power of r as H has, multiplied by an additional factor a/r: the latter part has the same inverse power multiplied by a factor $2\pi a/\lambda$, where λ is the wave-length corresponding to the period τ. As regards the

former part, the total kinetic energy of the radiation sent out beyond a distance r, being proportional to the integral of H^2 taken over the region outside the distance r and onwards to infinity, involves $a^2e^2\iint r^{-4}dSdr$, that is $a^2e^2r^{-1}$, so that there is no kinetic energy finally lost to the system on account of this term, although there may be oscillation of kinetic energy backward and forward within any finite range: as regards the latter part, the total radiation sent out involves in its sensible term a factor $(2\pi ae/\lambda)^2\iint r^{-2}dSdr$, which implies a uniform stream of radiation whose amount is a fraction of the order $(a/\lambda)^2$ of the radiation of one of the electrons by itself. The product term in H^2, being fluctuating, gives no flow of energy. Similar statements apply to the potential energy, which depends on the electric force. This comparison of the orders of magnitude of radiant effects applies in a general way to any doublet for which the vector $\Sigma e\dot{v}$ is null, or to a molecule involving any number of revolving electrons provided the vector $\Sigma e\dot{v}$ is null for it: it asserts that the radiation of such a molecule is less than that of one of its electrons by itself, moving with the same speed, in the order of the square of the ratio of the diameter of the molecule to the wave-length of the radiation. The radiation actually emitted by molecules of matter has wave-lengths of, at the very least, 10^3 diameters of the actual molecule: thus in such a case the result of this mutual interference between the radiating electrons is that the actual radiation is less than (on our view that the diameters of electrons are very small compared with their distances apart in the molecule, enormously less than) 10^{-6} times what would be the aggregate of the radiations of its various electrons all moving independently*.

155. Thus, assuming merely that an ordinary material

* This argument may be compared with Sir George Stokes' explanation of the interference between the front and rear of the section of a vibrating telegraph wire, which effectually prevents communication of sound from it directly to the air. Cf. Rayleigh, 'Theory of Sound,' ii, § 324.

molecule involves in its constitution rapidly moving electrons, a hypothesis which is at present very widely if not universally entertained, we have ascertained that the condition that must hold to avoid the frittering away of the internal or constitutive energy of such a molecule by radiation is that the vector sum $\Sigma e\dot{v}$ shall be permanently null. It has been already noticed that this condition is not satisfied by a simple free doublet composed of a positive and a negative electron revolving round each other: such a doublet is in fact a powerful radiator (§ 149) of the same type as a Hertzian oscillator, and so could not constitute a molecule. But it is easy to imagine steady systems for which the condition is satisfied, for example a ring of three or more positive electrons revolving round an inner ring of negative ones which is also revolving. The question of stability for such illustrative groups would afford extensive and interesting mathematical developments. The condition for absence of radiation is of course satisfied for every motion of translation of a chemically saturated molecule, because for such a system Σe is null.

The very striking fact that the wave-lengths of free radiant vibrations of molecules are such large multiples of their diameters has always invited explanation. On a statical conception of a molecule, or rather the common one which compares its vibrations to those of a statical system like a spring, this fact would suggest a very slight spring very heavily loaded. On the dynamical conception here employed, it involves that the orbital velocities of the electrons are of about the same order of smallness (exceeding 10^{-3}) compared with the velocity of radiation as are the molecular dimensions compared with the wave-lengths. The present analysis suggests a reason, in that the energy of orbital groups moving with greater speeds would be through time sensibly dissipated by radiation, so that such groups could not be permanent. This explanation is based on, and also required by, the mere hypothesis that molecules carry electrons: the further question whether they are wholly constituted of electrons is not here involved.

156. It has already been seen that there is no sensible

time-lag in the electric inertia of an electron owing to time being required to re-establish the steady field in the surrounding aether when the motion of the electron changes, the reason being that the energy, kinetic and potential, of the whole of that steady field, whose magnetic and electric vectors both vary inversely as the square of the distance, is practically within a distance of say 10^2 diameters of the nucleus of the electron, over which the field is established in an excessively small time. By an argument similar to that of § 148 it appears that for a molecule for which the vector Σev is null (and therefore also the vector $\Sigma e\dot{v}$ null), the energy is far more concentrated even than this. In either case therefore, when we consider that the diameter of the nucleus of a single electron is a small fraction of the diameter of the molecule, it appears that the constitutive part of the energy is to all intents *in* the molecule itself and only very minutely and residually in the surrounding field. It has here in various connexions been suggested that the electric inertia involved in the kinetic part of this energy is the total material inertia of the molecule: to deny this is in effect to say that the energy of motion of the molecule is not all in the electro-optic aether, and to raise the question where then it can be. Such an hypothesis does not preclude the possibility of as much structure in the molecule as may be required, for example in the wider relations of chemistry; but it asserts that this structure is entirely of the nature of aethereal constraint or permanent interlocked strain-configuration in the aether, and that it does not involve modes of transmission of disturbance from part to part of type other than the aethereal one here discussed.

In connexion with the explanation of § 155, it will suffice merely to mention how much is effected towards rendering the kinetic theory of gases consistent and intelligible, when we can have gaseous molecules which will not be set into violent radiation at each mutual encounter.

CHAPTER XV

ON THE NATURE OF ORDINARY RADIATION, AND ITS SYNTHESIS INTO REGULAR WAVE-TRAINS

The Röntgen Radiation

157. THE Röntgen radiation has been from the first ascribed to violent aethereal disturbances, set up by the impacts of the rapidly moving particles of the cathode streams against the walls of the vacuum-tube in which they travel. According to Sir George Stokes this radiation is composed of thin spherical sheets of disturbance sent out into the aether by the sudden impacts, in part arising from the shocks imparted to the molecules forming the walls of the tube, and in part from the arrested cathode particles themselves. In so far as these sheets of radiation are due to sudden but transient disturbance of the electrons in the molecules of the walls of the tube, the magnetic force belonging to them alternates in direction in crossing each thin shell of pulse so that the average value taken across it is null. In so far as they are due to the sudden arrest of the cathode particles, each of which involves (if it is not altogether constituted by) a moving electron, this balanced alternation of magnetic force across the thickness of the sheet does not hold: the force may be in the same direction all the way across*. As during the progress of the impact the accelerations of arrested cathode particles and of the disturbed electrons of the tube-wall will be presumably of the same order of magnitude, we would naturally conclude, from the formula expressing the radiation in terms of the acceleration of the electron (§ 150), that

* This distinction has been pointed out by Godfrey, *Proc. Camb. Phil. Soc.* 1898.

these are both concerned in the emission of radiant energy. In addition to the thin pulse arising from the sudden shock imparted to the molecules of the tube-wall, we would expect to find also more continuous radiation due to their state of vibration which would ensue: this would be represented by the phosphorescent light which accompanies the phenomenon; and in part, it might be thought, if very high free periods are sufficiently predominant, by the Röntgen radiation itself, for that radiation has no properties that may not be ascribed to ordinary continuous trains of radiation, of period high enough to be beyond the influence of vibrations of such periods as are readily excitable in material atoms. This consideration however strikes both ways, and in fact tends to show that the rays cannot consist of *definite* radiation arising from very high definite free periods of the atoms, because it would then excite those same periods and thus be subject to refraction. We have thus to fall back on the view that the Röntgen rays consist of irregular non-periodic radiation due to general disturbance. This indeed would also be the case for ordinary non-selective radiation from an incandescent solid or liquid, according to the explanations advanced by Lord Rayleigh: perhaps the most striking phenomenon in physics, whether in its theoretical or experimental aspect, is the way in which a prism or grating separates out a formless tumultuous mass of radiation advancing on it into a series of trains of regular undulations.

The very high velocity of the cathode particles, being a considerable fraction of that of radiation, must produce disturbances on impact much more sudden and intense than the mutual encounters and decompositions of the atoms of an incandescent body could initiate: hence, bearing in mind the law that the intensity of the radiation depends at each instant on the square of the acceleration of the radiating electron, and the circumstance that its penetrating power depends on the abruptness of the disturbance, the actual characteristics of the Röntgen rays are such as would be expected.

If we assume that a cathode particle impinging with velocity v is reduced to rest in a distance comparable with the linear dimensions l of a molecular orbit, we can compare very roughly

the intensity of the Röntgen radiation from an arrested electron with that of the steady radiation of an isolated electron describing such an orbit with velocity v': the ratio of the energies is of the same order as that of the squares of their accelerations, which is $(v/v')^4$: so that the Röntgen radiation is much more intense, while it lasts, than would be the ordinary radiation of such an isolated electron, it being assumed on grounds stated above (§ 155) that the latter does not describe its molecular orbit with velocity comparable with the speed of radiation.

158. The question arises how thin a radiant pulse must be in order that it may get across a material medium without exciting a sensible amount of vibrational energy in the molecules of the medium, and therefore without being subject to scattering sideways, as regular refraction is not to be expected in such a case.

The absence of diffraction of the Röntgen radiation is connected by Sir G. G. Stokes with the thinness of the pulse, and with the additional circumstance that, as presented to his view, it is a back and forward pulse containing in its thickness as much negative as positive displacement. For the part of the radiation that arises from the sudden stoppage of the cathode particles we have seen that this back and forward character will not hold: and the absence of diffraction will for this part be conditioned by a less potent, but still sufficient cause**, the thinness of the pulse. The circumstances of the other case, contemplated by Sir George Stokes, would be quite analogous to those of a train of waves whose wave-length is of the order of the thickness of the pulse.

When a very thin pulse of this kind traverses a material medium, the effect it produces on each electron is nearly the

** In this case the elements of the wave-front would send out disturbances of concordant phase, which would all reinforce instead of cancelling each other at places inside the geometrical shadow: thus in one sense there is copious diffraction. But the aggregate disturbance would not travel into the shadow as an abrupt pulse: owing to the different distances of the sources of its constituent elements it would be drawn out inside the shadow into a disturbance of gradual character, which would not possess the characteristic properties of an abrupt pulse but rather those of ordinary light. For a mathematical investigation, cf. Sommerfeld, *Physikalische Zeitschrift*, Nov. 1899.

same as if that electron were isolated from its surroundings: the pulse imparts a sudden impulse to each electron and then has passed on. The interactions of the electron, maintained through the aether, with other electrons in the same molecule, being elastic forces depending on strain, will remain finite, and so will not be in a position to ease off materially the velocity communicated by such an impulsive action. This is in contrast with the behaviour of a train of ordinary radiation arising from a regularly vibrating molecule: such a wave-train gradually establishes by resonance, in the course of a large number of periods, a state of molecular vibration involving all the electrons in the affected molecule in a connected manner as a single dynamical system. In the first case there is loss of energy by communication from the pulse to the electron, involving absorption of the radiation, which is simply proportional to the number of electrons per unit volume, irrespective of their molecular arrangement*. But here a real distinction arises according to whether the pulse is a back and forward one, or one preponderating in a given direction such as would result from the sudden stoppage of a cathode particle: a forward impulse immediately followed by an equal backward one will communicate but little energy to an electron which lies in its path, as compared with an impulse in which one side preponderates: thus the kind of Röntgen radiation that comes directly from the sudden stoppage of cathode particles should be less penetrating than the kind which comes from the shock to the electrons that belong to the molecules of the walls of the tube. In the case of a regular wave-train, on the other hand, absorption arises owing in part to molecular decompositions, and in part—in gases perhaps principally—to change of molecular orientations arising from mutual encounters. It was found by Röntgen himself, and has been verified by subsequent observers, that the order of opacity of substances for Röntgen rays is roughly the order of their densities. Various considerations impart some plausibility to a theory that the inertia of matter is simply the electric inertia of

* Considerations in some respects similar to the above were advanced by Sir George Stokes in his Wilde Lecture, *Proc. Manchester Lit. and Phil. Soc.* 1897. Cf. also § 163 *infra*.

the electrons which are involved in its constitution: this law of absorption seems to furnish an experimental consideration tending in the same direction.

On the nature of the analysis of ordinary Radiation

159. For simplicity and definiteness, let us begin with an illustration furnished by plane waves of sound excited in an infinite straight pipe of uniform section. Suppose the wave-train to be originated from a condition of constrained equilibrium in which a length of the air in the pipe is held in a state of uniform compression, while the remainder is in its natural state: when this constraint is released two simple pulses of compression start from the compressed region, one carrying half the compression forward and the other carrying the remaining half backward, each with the velocity of sound in air. Now consider one of these travelling pulses by itself. At any instant the Fourier analysis will resolve the compression into an infinite series of simple harmonic compressions, of all possible wave-lengths, and each filling the whole tube to infinite distance in both directions. Each of these components travels onward as a simple wave-train of unlimited length, and *if it existed by itself* would be recognized as such by an ear or other suitable acoustic receiver. But this comes near to saying that a single pulse of limited dimensions travelling through the quiescent air involves an infinite series of regular waves travelling both in front of it and behind it, although in neither of these positions is there really any disturbance of the air at all. Yet in the perception of tone (and of colour in optics) an objective validity is assigned to this kind of Fourier resolution. The explanation arises from the circumstance that, for purposes of perception, it is the sequence of disturbance at the place occupied by the receiver that is analyzed into time-periodic constituents in the Fourier manner, of each of which the receiver takes separate account. The analysis into simple wave-trains in space is not an objective one, and possibly serves no useful purpose so long as the transmitting medium is uniform: in the theory of refraction, or rather that of dispersion, such an analysis however constitutes

an essential part of the machinery of mathematical discussion. But its character is only formal: in order to obtain the objective result of the dispersion, a receiver is stationed at some definite place in the medium and takes account of the course of the total disturbance which goes past that place as time proceeds.

159**. To illustrate the difference between the theoretical analysis of a disturbance by Fourier's theorem and its objective analysis by a receiver on which it falls, when the disturbance has no periodic features, the case above described may be further considered. The receiver may be there constituted by a piston enclosing a cushion of air in a cylinder: a complex receiver such as might roughly illustrate the action of an ear or eye, may be considered as formed by an aggregation of such independent elements, with different free periods. The equation of vibration of one of these receiving elements will be

$$\ddot{u} + 2k\dot{u} + n^2 u = U,$$

where U represents the force which the piston experiences and is a function of the time. The solution for any form of U may be derived symbolically, D representing d/dt, as follows, n'^2 denoting $n^2 - k^2$ so that $2\pi/n'$ is the free period:

$$\begin{aligned} u &= (D^2 + 2kD + n^2)^{-1}\, U \\ &= (2\iota n')^{-1}\left\{(D + k - \iota n')^{-1} - (D + k + \iota n')^{-1}\right\} U \\ &= (2\iota n')^{-1}\left\{e^{(-k+\iota n')t} D^{-1} e^{(k-\iota n')t} U - e^{(-k-\iota n')t} D^{-1} e^{(k+\iota n')t} U\right\} \\ &= n'^{-1} \int_{-\infty}^{t} e^{-k(t-t')} \sin n'(t-t')\, U' dt', \end{aligned}$$

on writing t' for t under the sign of integration. If the time of action of the force U extend sufficiently far backwards, the effect of the initial circumstances will have faded out. Thus

$$u = n'^{-1} e^{-kt} \left\{ \sin n't \int_{-\infty}^{t} e^{kt'} \cos n't'\, U' dt' - \cos n't \int_{-\infty}^{t} e^{kt'} \sin n't'\, U' dt' \right\}.$$

This expresses the motion as two simple harmonic vibrations, of the period and damping belonging to the receiver, but with changing amplitudes. To determine the amplitudes at any

instant, we may take that instant as the time $t=0$: thus they are obtained by a process, of the Fourier type, of sifting out the constituent elements of that period in U throughout past time, but modified by the presence of a damping coefficient which renders remote times inoperative.

In the present example U is constant during the time 0 to τ taken by the compression to pass over the piston, supposed initially at rest. Thus at any instant after the compression has passed over, we have if $\tan\alpha = k/n'$,

$$u = \frac{1}{n'}\int_0^\tau e^{-k(t-t')}\sin n'(t-t')\,Udt'$$

$$= -Un'^{-2}\cos\alpha\,\{e^{-kt}\cos(n't-\alpha) - e^{-k(t-\tau)}\cos(n't-n'\tau-\alpha)\}.$$

Hence in this case there is an unlimited uniform damped train of vibrations of the natural period of the receiver, followed after time τ by an equal train with the disturbance reversed. When there is no damping in the receiver so that k is null,

$$u = 2n^{-2}\,U\sin\tfrac{1}{2}n\tau\sin n(t-\tfrac{1}{2}\tau),$$

which represents the interference of these two trains of vibrations.

Thus each constituent of the complex receiver executes simple trains of vibrations of its own period, endless or gradually damped as the case may be, but those constituents whose free periods range near certain values respond most strongly. If the damping were very slight, the constituents whose free period is equal to τ or a submultiple of it would be undisturbed, and those of intermediate periods τ' equal to the roots of the equation $\tan\dfrac{\pi\tau}{\tau'} = \dfrac{1}{2}\dfrac{\pi\tau}{\tau'}$ would be most excited.

On the other hand when the incident disturbance is a train of waves, as it becomes longer and more regular the response to it approximates more and more to a regular forced vibration of the period of the disturbance, but most intense in those constituent receivers whose free period is nearest unison with it.

This point may be further illustrated by the solution corresponding to a damped train of existing radiation represented by $U = U_0e^{-pt}\cos qt$ lasting from its sudden beginning when $t=0$ until it decays to insensible amount: then the solution corresponding to an initial state of equilibrium is

$$u = U_0 n'^{-1} \int_0^t e^{-k(t-t')} e^{-pt'} \sin n'(t-t') \cos qt' dt'$$

$$= \tfrac{1}{2} U_0 n'^{-1} [\{(k-p)^2 + (n'-q)^2\}^{-\frac{1}{2}} \{e^{-pt} \cos(qt-\alpha) - e^{-kt} \cos(n't-\alpha)\}$$
$$+ \{(k-p)^2 + (n'+q)^2\}^{-\frac{1}{2}} \{e^{-pt} \cos(qt+\beta) - e^{-kt} \cos(n't-\beta)\}],$$

where $$\tan \alpha = \frac{k-p}{n'-q}, \quad \tan \beta = \frac{k-p}{n'+q};$$

or say

$$u = Ae^{-pt} \cos(qt-\gamma) + Be^{-kt} \cos(n't-\delta),$$

as might have been obtained by more direct methods. Hence in this case two regular trains of vibrations are started instantaneously in the receiver, one of the period of the exciting wave-train, and the other of its own free period. The first decays according to the same law as the exciting train, the second according to the same law as a train of free vibrations of the receiver. Both trains of vibration are of the same order of magnitude at starting, as regards amplitude. The important case is that in which the period of the influencing train is nearly the same as that of the receiver: then if the damping coefficients are both small, *or else nearly equal*, the initial amplitudes of both trains of vibrations are large.

It appears from this solution, which corresponds to a fair representation of actual conditions of resonance, especially in optical applications, that when the natural vibrations of the receiver are damped more quickly than those of the exciting wave-train, the receiver vibrates mainly in the period of the latter: but when the exciting train is the more rapidly damped, the vibrations of the receiver are mainly of its own period.

Considerations of this kind have scope in the analysis by resonators of the long-period radiation from a Hertzian electric oscillator. If we imagine a conductor on which an electric charge is held in a disturbed condition by constraints, this involves a state of disturbance of the intrinsic aethereal strain all round the conductor: when the constraint is released this strain subsides into its equilibrium state corresponding to the charge on the conductor, gradually when the removal of con-

straint is slow, but in part by wave-propagation when it is sudden. The case is then analogous to that of a load suddenly removed from a portion of an infinite elastic solid, or to the sudden elevation of a solid dipping in a large expanse of liquid: a pulse of waves travels out from it which is devoid of periodic character, and the period of a receiver that most strongly responds to it depends on the constitution of the receiver as well as on the character of the incident pulse. It is possible to have definite period in the radiation from an electric vibrator only when it involves an outer reflecting envelope to entirely prevent loss by radiation, or when the supply of potential energy is partially protected by adiabatic boundaries from immediate dissipation, as for example is the case in a charged condenser, or when its vibrations are maintained by external agency.

In an ordinary Hertzian oscillator the disturbance emitted cannot be of this dead-beat character, otherwise a resonator would not respond in any definite manner. But it is very rapidly damped out; yet if the damping is regular, the analysis above given shows that the resonator is in general most effective when it is most nearly in unison with the incident wave-train.

160. The radiation from an incandescent solid or liquid, though not from a gas, presents as a whole nothing of a periodic character, for it arises from the independent and irregular disturbances of countless molecules; it thus has the appearance of a formless tumultuous mass of radiant disturbance, advancing with the velocity of light. Yet this need not imply that periodic qualities are wholly absent in it.

The theory of optical dispersion elaborated by Cauchy made that phenomenon depend on simple statical discreteness of the transmitting medium: such discreteness exists, is a *vera causa*, in the form of the molecular structure of matter; but it has been pointed out by various writers that this molecular structure is too fine-grained to produce more than a very minute fraction of the dispersion actually occurring. It is thus necessary to base dispersion on sympathetic oscillation of the

material molecules, instead of their mere inertia: nor is this mode of explanation an afterthought, for to the acute mind of Young, who was the very first to attempt physical theories of these actions, it presented itself as the natural way in which the material molecules would influence the waves of radiation. But if sympathetic vibration in the molecules is to be effective, there must be regular periodicity somewhere in order to excite it: and in order to excite it in the very exact and definite way that is put in evidence by the extreme refinement of the analysis of light by prisms, this periodicity must last without discontinuity over a large number of vibrations. What then is its source?

The absorption spectra of partially transparent solid and liquid materials indicate more or less roughly a preference of their molecules for vibrations around certain periods: and it might at first appear that we must assume in the case of solids as well as gases that each of their molecules is in the main in a state of true steady vibration continued over a fairly large number of periods, during the intervals between the successive disruptions or disturbances of chemical (or electric) bonds that are the exciting cause of its radiation. The aggregate of the radiation sent out by all the molecules, being devoid of any relation between the phases or durations of these free trains of vibration, would be of sensibly periodic character, but gradually changing in form as the contributions from individual molecules change. In a theoretical treatment of the question of dispersion the continuous wave-train which is the subject of the mathematical analysis may in fact be taken as the wave-train emanating from a single molecule of the radiating source: from the solution of the problem of propagation which holds for it we would pass to the general solution by simple addition or superposition of disturbances. In the result there would at any instant be a definite amplitude and phase, but these would be gradually changing, so that in a time infinitesimal compared with the duration of a visual impression they would be totally different. Although, therefore, there would then be nothing observable of the nature of absolute phase in the aggregate, even in a plane-polarized train of light

of definite wave-length, yet we could make observations as to change of phase in such a train, on reflexion, for example, from a metallic medium: each of the independent periodic trains, from the individual molecules, of which the radiation is composed would then undergo the same change of phase; and as the observable result is the same for all these subtrains it will persist on their superposition, and may be detected by the elliptic polarization that can be derived from it in the well-known ways. In any other sense than this the phase of the vibration of an actual beam of light would be illusory. In all such cases of random phases the energy carried in any sensible time by the aggregate of the radiation is the sum of the energies carried by the simple wave-trains of which it is constituted.

But an explanation of this kind is vitiated for the case of the superficial radiation from incandescent solid and liquid bodies, by the circumstance that if there were any underlying periodicities in the radiation, they would reveal themselves in bright lines, more or less distinct, when it is analyzed by a prism or grating. We must infer that the radiation from an incandescent solid or liquid is wholly devoid of periodicity, that the interval between successive disturbances of each radiating molecule is too short compared with its free periods to allow any regular train of vibrations to come into existence. It is different in the case of rarefied gases, and to some extent in the case of other bodies radiating from their interiors.

161. The dynamics of refraction and dispersion of ordinary white light must therefore be developed on other lines. The solution of the apparent paradox that is involved is best brought out by considering a simpler question. How is it that, if white light is wholly devoid of even latent periodicity, a small nearly homogeneous portion separated out from it by prismatic dispersion shows regularity reaching over a large number of waves, as is evidenced by the fact that with it a succession of interference bands may be produced, whose number can be made quite large when the light is sufficiently homogeneous in refractive index?

As a preliminary question, not involving the intervention

of dispersion, how is it that a split beam of light, belonging say to the thallium green line, shows, when the two parts are reunited after traversing paths of different lengths, a succession of interference bands up to the order 10^5 in number?* The train of radiation belonging to the thallium line is the aggregate of subtrains emitted by the different molecules of the vapour, in this case freely vibrating in the intervals between the molecular encounters. Each of these subtrains starts more or less suddenly, proceeds for a time, and then stops more or less suddenly, or is transformed, at the next encounter; and there is no relation whatever between the phases of the various subtrains. When all these are superposed a resultant radiation is obtained, whose graphical representation is a curve sensibly periodic for a number of vibrations comparable with 10^4, but in which over a longer range the amplitude and phase gradually change in a wholly fortuitous manner. Hence if we superpose two lengths of this radiation, close enough together and in opposite phases, they will cancel each other as regards energy of illumination; and alternations will thus appear when the two portions are made to slide, so to speak, along each other**. If however we compare two lengths along the ray which are so far apart that none of the regular molecular subtrains that constitute the one have persisted on into the other, there cannot possibly be any relation between the characters of the disturbance along these portions. It appears to follow there-

* Cf. Mascart, 'Traité d'Optique', i, p. 178.

** One is liable at the first glance to conclude that the resultant of a very great number of simple wave-trains of equal periods and about the same amplitude, but with phases arbitrarily distributed, is of magnitude comparable with that of one of them, and so very small: if that were so the addition of a single new constituent would entirely alter the vibration, and there would be no regular resultant motion at all. Supposing for simplicity the constituent harmonic motions at a point to be all in the same direction, each may be represented by a radius vector of length equal to its amplitude and inclination equal to its phase: although the vectors are of the same order of length, and distributed indifferently all round the origin, all that we can conclude as to their resultant is that its amplitude is at any instant small compared with the sum of their amplitudes. The means of the energies, proportional to the squares of the amplitude, are in such a case additive: hence for the resultant of n equal vibrations of arbitrary phase, the most probable amplitude is of the order $\sqrt{n}$ times that of a constituent. Cf. Rayleigh, *Proc. Math. Soc.* May 1871; *Phil. Mag.* Aug. 1880; 'Theory of Sound,' ed. 2, § 42*a*.

fore that in a case like this, in which no dispersion artificially produced by prisms or gratings is taken advantage of, the fact that 10^5 successive interference bands can be counted is evidence that a considerable proportion of the molecules have gone on vibrating regularly for that number of periods, in other words that there is time for about 10^5 vibrations of the light of this thallium line between successive disturbances of the vibrating molecule*. The interval between its successive encounters, on the theory of gases, is in fact of this order: although ordinary gaseous encounters cannot start radiation, yet by altering the orientation of the molecule they can introduce discontinuity.

162. These considerations however still leave unexplained the origin of the periodicity which dispersion introduces into the constituents of ordinary light. The answer, as given by Gouy† and by Rayleigh‡, is that it is constituted by the optical apparatus which produces the dispersion.

When a grating is employed, the radiation which appears in any given part of the spectrum is made up of contributions from each line or physical element of the grating; thus in the case of the first spectrum it is an average struck over a length equal to as many of its wave-lengths in the incident radiation as there are lines, say N, in the grating. If then we compare two portions of this selected radiation that are less than N wave-lengths apart, they will have constituents in common, and if they are less than $\frac{1}{2}N$ wave-lengths apart, they will have more than half of their constituents in common. We should expect then that in such a case the number of successive interference bands that could be counted, independently of any original periodicity in the light, would be of the order of $\frac{1}{2}N$.

If the dispersion is produced by prisms instead of a grating, the explanation must be on different lines. The essence of the

* This is not quite exact: to isolate the thallium line a grating (or prism) must be used, which will increase the number of possible correspondences by about half the total number of its rulings: but this correction will be negligible compared with 10^5 unless the grating is a powerful one.

† 'Sur le mouvement lumineux,' *Journal de Physique*, 1886.

‡ 'Wave Theory,' *Encyc. Brit.*, 1888; *Phil. Mag.* June 1889. [Cf. also Schuster, *Phil. Mag.* June 1894, and *Comptes Rendus*, 1895, for a detailed discussion.]

action of a prism is that the light which has traversed it towards the thicker side is delayed; in this case (not in the previous one) when all the parts are brought finally to a focus they arrive there at the same time, by Fermat's principle, and they thus all belong to the radiation emitted at the same instant from the source. The explanation must therefore now lie in the dynamics of refraction. As already explained, refraction must arise from the interaction of sympathetic vibrations induced in the molecules by the radiation passing across them: this requires periodicity, and the difficulty was as to whence that periodicity came in the case of white light. In the mathematical analysis of dispersion the radiation is supposed to be sifted into regular harmonic components by Fourier's theorem; each of these components is transmitted independently, inducing its own sympathetic vibrations in the molecules. It is then the nature of the Fourier analysis that should indicate the source of the periodicity of the dispersed light; when this analysis is expressed in the following form the characteristics that are being sought for will appear.

Any function $f(t)$ however complex, provided it has only a finite number of singularities within the range considered, can be resolved, in the interval between the values $-\tau$ and $+\tau$ of the variable, into a series of simple harmonic functions of multiples of $\pi t/\tau$, there being two of these functions for each multiple r, differing in phase by a quarter period, namely $\cos \pi rt/\tau$ and $\sin \pi rt/\tau$. The amplitude of each is found by a process which amounts to taking an average, over the range considered, of the amplitudes of all partial series of harmonic sequences of that type that exist in it. The nature of the process, so far as we are concerned with it in physical problems of vibration, in which damping agencies are always present, will appear when we pass towards the limit of τ very great when there would be a continuous distribution of components of all periods in the result. We have now $f(t)$ given for a long time past and future: we may suppose for definite illustration that it represents the aggregate, at a given point of space, of the radiation from a system of molecules say of a gas, and that each of these molecules contributes, owing to its free

vibrations, uniform subtrains of period $2\tau_0$, a submultiple of 2τ, each beginning and ending abruptly. Let one of these incomplete subtrains be $a_r \cos \pi rt/\tau_0 + b_r \sin \pi rt/\tau_0$, lasting for n_r periods: then in the Fourier representation there will be a complete train $A \cos \pi rt/\tau_0 + B \sin \pi rt/\tau_0$, extending through all the time, say N periods, where $A = \Sigma n_r a_r/N$, $B = \Sigma n_r b_r/N$, together with trains usually much less important of the other harmonic periods. This indicates the character of the process of mathematical resolution into harmonic components: in the general case when $f(t)$ is not made up of such periodic subtrains the process retains the same character, but cannot be expressed so simply. Now to carry out this process with mathematical strictness we require to know the form of $f(t)$ for all time††, and to average over all time: but in a physical application all time merely means a time sufficiently long to cover all the effective factors of the phenomenon. The Fourier analysis preparatory to the dynamics of refraction of compound radiation thus consists of the culling of contributions from a long series of equal lengths before and after, to make one wave-length: and the periodicity is thereby theoretically manufactured just in the same manner as it would be practically made by a grating. Of course the object of the theoretical Fourier analysis is only to facilitate the representation of the dynamical action of the prism: it is the prism that actually separates out, by the agency of the sympathetic molecular vibrations causing its dispersion**, the formless incident stream of radiation into harmonic wave-trains. In this case it is not so easy to assign an upper limit‡‡ to the number of consecutive interference bands to be expected for very small breadth of slit as it is when a grating is used: the Fourier resolution involves no limit of that kind:

†† In passing to this limit, where there is a continuous distribution of Fourier periods, the origin of the aggregate of Fourier elements whose periods are included within small assigned limits is what is to be traced: to do this rigorously would be a long affair: but the very indefinite indications above given may perhaps suffice for the present purpose in which there is no practical possibility of exact formulation.

** This may be illustrated more directly from the general solution on p. 240.

‡‡ Namely, the limit, depending on the material of the prism, beyond which increase of resolving power, estimated geometrically, is no longer effective.

the limit should rather be connected with the average undisturbed duration of a train of sympathetic vibrations in the molecule (or a molecular complex) † of the refracting medium, as well as possibly in part with the degree of coarseness of its molecular structure.

163. The radiation from an incandescent solid or liquid source is thus different in kind from the selective radiation of a gas. Its most striking feature is that the range of periods in its continuous spectrum spreads in a definite manner towards the upper end of the spectrum as the temperature rises. This perhaps is related to the fact that the superficial molecular disturbances, which in the main emit the radiation, increase in abruptness with rise of temperature.

The carriers which constitute the cathode rays in a Crookes' vacuum tube are known to travel at speeds not far short of the order of that of radiation. Their collision with an obstacle is therefore of excessive abruptness; and we should thus, even *a priori*, anticipate a large admixture of very high periods in the Fourier analysis of the Röntgen radiation thereby originated. It is conceivable that a very fine diffraction grating, of an ideal substance sufficiently opaque to the Röntgen radiation, might do something towards its resolution into spectra: but the opacity would have to be very great on account of the short wave-length.

Suppose there is a system of free single independent electrons (like electrolytic ions) existing in the course of a train of regular radiation of definite period: the transverse electric force in the waves would set them vibrating in unison and so would convert each of them into a radiator: the positive and negative ones would have their vibrations in opposite phase and so would both be radiating in the same phase which would be a quarter of a period behind the exciting radiation: thus positive and negative solitary electrons would not interfere with each other in the direction of diminishing the total radiation of energy, but the contrary. Contrast this with the case in which

† This resonance has been strikingly compared by Sir George Stokes (*loc. cit.*) to that of the sounding-board of a pianoforte.

it is groups of combined electrons that lie in the path of the radiation. The strength of their mutual connexions is indicated by the high frequency of the free periods of the molecules which they constitute: if the period of the incident radiation is comparable with these periods, its electric force can do practically nothing in the way of pulling the positive and negative electrons of the molecule independently in different ways, but will merely induce a state of connected vibration, which if it can simultaneously radiate to an appreciable extent, will involve a drain of the energy of the exciting radiation after the ordinary manner of selective absorption. But now consider the incidence of radiation of a period much higher than any of the main free periods of the molecule: this will affect each electron of the molecule more or less independently because the elasticity of the molecule has not now time to get all the electrons under control on account of the rapidity of the alternations: thus each electron will practically be an independent vibrator just as if they were all free. The absorption will now depend on the aggregate number of electrons per unit volume, but hardly at all on their molecular aggregation: if we adopt the view that the electric inertia of the electrons is the whole of the inertia of matter it will follow that the absorption of radiation of very high period tends to become proportional to the density of the matter present and to nothing else, which is in a general way the state of affairs with regard to the Röntgen rays.

APPENDIX A

ON THE PRINCIPLES OF THE THEORY OF MAGNETIC AND ELECTRIC POLARITY: AND ON THE MECHANICAL SIGNIFICANCE OF DIVERGENT INTEGRALS.

1. THE physical phenomenon of polarity is most simply illustrated by magnets; in them the polar character has long been known to be present in their smallest parts, so that the physical element is bipolar instead of being a single attracting pole. Accordingly the first mathematical development of this subject was contained in Poisson's theory of magnetization by influence. All the main theoretical relations of polarized media were there deduced from a hypothesis of mobile magnetic matter acting directly across a distance, which though physically unreal supplied a picture of the relations of the actual phenomena which was sufficient for the purpose in view. The subject however hardly got a start as a physical theory until the appearance of Lord Kelvin's memoirs on 'A Mathematical Theory of Magnetism' ‡ in 1849: the treatment there given added little to Poisson's results, but it cleared away the artificial hypotheses and aimed at evolving the theory solely in terms of phenomena that could be observed and quantities that could be measured, making use of the principle of negation of perpetual motions and of the methods of the conservation of energy: it is in the physical conceptions, referring not to molecules but to media treated as continuous,

‡ Cf. Reprint of Papers on Electrostatics and Magnetism, pp. 344 sqq.

which are there defined and developed in their mutual relations, that the physical theory of polarity took its rise.

Shortly before this time Faraday had rediscovered the fact, already known to Cavendish in the previous century, that the nature of the dielectric medium plays a prominent part in the transmission of electric influence between one charged body and another, a fact which at first much puzzled the school of mathematical theorists who were more familiar with astronomical attractions at a distance than with intermolecular actions. In the memoirs above referred to, Lord Kelvin, referring back to his paper 'On the Elementary Laws of Statical Electricity'‡ where he had explained for the first time the nature of this dielectric action on the analogy of Poisson's theory, emphasizes the fact that "however different physically, the positive laws of the phenomena of magnetic polarity and dielectric polarity are the same, and belong to a very important branch of physical mathematics which might be called 'A Mathematical Theory of Polar Forces'." This view of dielectric action was also treated at length not much later by Mossotti and others.

2. The next great step in advance in electrical science was Maxwell's ascription of electrodynamic properties to changing dielectric polarization. In order however to fit the phenomena of static electric discharge into the scheme which he had fashioned from a general survey of electrodynamic phenomena, which almost required on grounds both of theoretical simplicity and physical probability that all electric currents should flow in streams round complete circuits, it was necessary to assume that dielectric polarization was itself a stream vector. On the theory of Poisson and Kelvin the polarity induced in the material medium does not however possess this property: so Maxwell virtually ignored that theory and postulated some new kind of effect which he called dielectric displacement, which in isotropic media is equal to the electric force multiplied by $K/4\pi C^2$, with such generalization as is natural for the case of crystalline media, and which is defined solely by its circuital

‡ *Camb. and Dub. Math. Journal*, Dec. 1845; *Reprint*, § 447.

or stream property. Thus, (P, Q, R) being the electric force, in an isotropic medium

$$\frac{d}{dx}\left(\frac{K}{4\pi C^2}P\right)+\frac{d}{dy}\left(\frac{K}{4\pi C^2}Q\right)+\frac{d}{dz}\left(\frac{K}{4\pi C^2}R\right)$$

is postulated to be null, except however at singular points in the medium from or towards which there is a convergence of the electric displacement whose total amount, when it is integrated all round the point, is called an electric charge—a *true* charge, as distinguished from the *apparent* charge which is only an aspect of polarization: these electric charges are postulated to be permanent singularities in the aether, involving intrinsic strain-configurations which can move about freely in that medium but cannot by any natural processes be created or destroyed.

This view of dielectric action was perfectly definite and precise, and exactly what was needed for general electric theory: but it was not a theory of polarity in the only known form, that of Poisson and Kelvin. The question naturally arose as to what it could really represent. Maxwell in his later methodical expositions tacitly declined to theorize on this subject: he had discarded the notion of action at a distance, whereas the theory of polarity as previously developed made use of direct attraction between the poles. When von Helmholtz took up the study of Maxwell's theory, this question became prominent: he naturally resolved to try what the known theory of polarity would lead to, and in consequence he had to deal with the effects of currents or displacements of electricity that did not flow in complete circuits: to form a basis for their treatment he generalized Neumann's expression for an electrodynamic potential energy between current-elements as far (or nearly) as it would bear without altering the laws for permanent currents flowing in closed circuits which were known to be correct. The result was his so-called extension of Maxwell's theory, which however, being based on distance actions, is in conception entirely foreign to Maxwell's view of transmission by a medium. He showed how by suitably adapting his constants, a scheme of equations formally equivalent, over a considerable range of theory, to Maxwell's could be obtained; while attempts,

continued through many years, to find out by experiment whether there were any phenomena that demanded a wider scheme than Maxwell's, culminated in Hertz's brilliant verification of the Maxwellian scheme in its simplicity.

3. The question however still remained as to the nature of Maxwell's dielectric displacement. The account of it that has been offered in the present discussion,—which agrees in its essential features with one elaborated on a different basis by Lorentz—is that it is of complex type: that it is in part a true polarization of the molecules of the medium, whose constituent electrons act on each other not by forces at a distance but by transmission of effect through the intervening aether, this part not being circuital or in any other way restricted as to form: that there is another part to be added which consists of true aethereal displacement, namely the aethereal strain that would remain if there were no matter present, this part not being electric flow at all: and that the sum of these two parts, of very different origins, forms a total displacement which is always circuital, and by virtue of the dynamical constitution of the aether (Chapter VI) has all the electrodynamic properties required in Maxwell's scheme. The only reservations that are to be made belong to the phenomena of radiation through moving material media, which have here been theoretically developed. This method, being to some extent the final synthesis of a theory which involves the laws and properties revealed by the analytical procedure of Maxwell,—and previously on the purely optical side by that of MacCullagh, who worked on a similar inductive plan,—builds directly on the dynamics of the aether whose constitution is effectively known, taking cognizance of the modifications involved in the presence of the electrons which form the mechanism of connexion between matter and aether.

4. The relations between the various vector quantities that occur in the general theory of polarity, which is thus fundamental in physical inquiry relating to continuous media, may be directly developed, with logical precaution and validity, in

the manner typified by the following synthesis of the relations connecting the magnetic force $\mathfrak{H}$ or (α, β, γ), the magnetic induction $\mathfrak{B}$ or (a, b, c), and the magnetization $\mathfrak{I}$ or (A, B, C), in the case of a body magnetically polarized.

Under static circumstances the usual argument, founded on the negation of perpetual motions, points to the magnetic force in regions outside the magnet being derivable from a potential. An expression for this potential may be obtained by integration from its value for a polarized differential element of volume. The ideally simplest type of a polar element is a doublet consisting of a positive point-pole m and an equal negative one separated from it by a minute distance l: such a doublet may formally be considered as produced by 'separation' of the positive and negative poles, which originally cancelled each others' effects by superposition, to the distance l apart in a definite direction: the potential arising from the doublet, at a distance r from it which is very large compared with l, is $Mr^{-2}\cos\epsilon$: this only involves M, equal to ml, called the moment of the doublet, and the angle ϵ which the distance r makes with the direction of this moment. It is verified at once that this moment is a vector quantity, because its potential is equal to the sum of those of its components when it is treated as a vector. When polarity is formally considered as introduced into an element of volume by the occurrence of any series of such separations of complementary point-poles, the total potential arising from it is, by addition, that due to a single vector which is the resultant of the moments of these component magnetic separations in the element. For an element of volume $\delta\tau$ this resultant moment may be denoted by $\mathfrak{I}\delta\tau$ or $(A, B, C)\,\delta\tau$ where the vector $\mathfrak{I}$ represents the intensity of magnetization, that is of magnetic separation, at the place. In a similar manner, in the theory of dielectrics the notion of displacement or separation of electric poles, that is of electric polarization, is introduced and defined. The physical origin of the polarity may of course be quite different from the ideal separation of poles by which it is thus formally represented.

5. At a point (ξ, η, ζ) outside the magnetic mass the

aggregate potential V arising from it is given by the integral

$$V = \int (Al + Bm + Cn)\, r^{-2} d\tau,$$

where (l, m, n) is the direction vector of the distance r of the point (ξ, η, ζ) from the element of volume $\delta\tau$ at the point (x, y, z) at which the intensity of the polarization is (A, B, C). Thus

$$V = \int \left(A \frac{d}{dx} + B \frac{d}{dy} + C \frac{d}{dz} \right) r^{-1} d\tau.$$

The point (ξ, η, ζ) being outside the polarized mass, the subject of this integration cannot become infinite; therefore transformation by integration by parts is legitimate; thus

$$V = \int (\lambda A + \mu B + \nu C)\, r^{-1} dS - \int \left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz} \right) r^{-1} d\tau.$$

Expressed in words, this means that the potential due to the polarized mass is at points outside it formally identical with that due to an ordinary volume-density ρ, equal to

$$-(dA/dx + dB/dy + dC/dz),$$

and surface-density σ, equal to the normal component of the polarity at the boundary, measured outwards: and the statical theory of polarity is thus reduced to the simpler theory of continuous distributions of attracting matter.

But at a point inside the polarized mass r^{-1} will be infinite for an element of the integral, and this transformation, considered as applied to a potential function which expresses the force by means of its gradient, is illegitimate. We shall still obtain a definite result for the force if we consider a point situated inside a cavity in the material, small in dimensions, yet very large compared with the scale of the physical polar element, that is, with molecular magnitudes; but now part of the surface-density σ will be on the wall of this cavity, and this part will give rise to a finite force (though not to a finite potential) at a point inside it, whose value depends on the form of the cavity and on the intensity of the polarity at the place. If we omit this *purely local part* of the magnetic force in the cavity, the remaining part, which is that

due to the polarized mass as a whole, will be derived from the general volume density ρ and surface density σ just as at an outside point. This latter part arising from the system as a whole, omitting the local term depending on the molecular structure at the point considered, is thus quite definite*, and is named the magnetic force $\mathfrak{H}$ or (α, β, γ); it is the gradient of a potential V, which is that due to densities ρ and σ as in the ordinary theory of the attraction of continuous mass-distributions.

It now follows by the fundamental theorem of Laplace and Poisson, relating to the attractions of continuous distributions, that, except at an interface of sudden transition,

$$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 4\pi\rho$$

where $$\rho = -\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right).$$

Hence, by transposition, we have everywhere

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0,$$

where $\mathfrak{B}$, or (a, b, c), is equal to

$$(\alpha + 4\pi A, \quad \beta + 4\pi B, \quad \gamma + 4\pi C).$$

Thus there enters into the theory this other fundamental vector $\mathfrak{B}$, named the magnetic induction, which has always the property of a stream, and which is connected with $\mathfrak{H}$ and $\mathfrak{I}$ by the vector relation

$$\mathfrak{B} = \mathfrak{H} + 4\pi\mathfrak{I}.$$

In analytical processes, the fundamental independent variables should be the ones about which most is already known; here they would be the magnetic force $\mathfrak{H}$ which is a gradient, and the induction $\mathfrak{B}$ which is a stream; from them would be derived, by means of the above relation, the polarization vector $\mathfrak{I}$ which is unrestricted as to form.

The intensity of polarity here denoted by $\mathfrak{I}$ may consist of a permanent part $\mathfrak{I}_0$ and an induced part $\mathfrak{I}_1$. The law of induction

* Cf. *Phil. Trans.* 1897 A, p. 233. The effort of Poisson to gain precision in this matter, occurring in the admirable summary of his theory in the introduction to the third memoir 'Sur la Théorie de magnétisme en mouvement,' 1826, is still instructive.

of polarity must be another relation connecting the induced part with either $\mathfrak{B}$ or $\mathfrak{H}$, whose actual form is to be determined by experiment, say it is $\mathfrak{I} = \mathfrak{I}_0 + f(\mathfrak{H})$; as $f(\mathfrak{H})$ is a definite function of $\mathfrak{H}$, the case of hysteresis, where there is no definite connexion between $\mathfrak{H}$ and $\mathfrak{I}$ independent of the past history of the system, is here excluded. For small intensities of the inducing force, the function $f(\mathfrak{H})$ will be simply proportional to $\mathfrak{H}$, so that $\mathfrak{I} = \mathfrak{I}_0 + \kappa\mathfrak{H}$; this, when expressed in terms of the fundamental vectors $\mathfrak{B}$ and $\mathfrak{H}$, assumes the form

$$\mathfrak{B} = 4\pi\mathfrak{I}_0 + (1 + 4\pi\kappa)\,\mathfrak{H}.$$

The mathematical analysis of induced magnetization now proceeds in the ordinary manner: in the equation expressing in terms of $\mathfrak{H}$ the condition of circuitality of $\mathfrak{B}$ the value of $\mathfrak{H}$ in terms of the potential V can be substituted, and a characteristic differential equation for V is thus obtained, which must in each particular problem be solved subject to the conditions of continuity of V and of the flux of $\mathfrak{B}$ across all interfaces.

It appears from the above that $\mathfrak{I}$, the moment per unit volume of the displacement of polarity, does not possess the stream property required for Maxwell's dielectric current in his electrodynamic theory; it is the other vector $\mathfrak{B}$, or rather $\mathfrak{B}/4\pi$ which is always equal to $\mathfrak{I} + \mathfrak{H}/4\pi$, that is in this respect the analogue of the total dielectric displacement of that theory.

The electric theory is also in one respect more general. In magnetism only simple polarity is involved arising from separation of complementary poles which is confined to the molecule: but in electricity there can also be distributions of single poles, infinitesimally smaller in total amount, constituting densities of *free* electrification. If we imagined a volume density of free magnetism ρ_1, it would add to the ideal volume density ρ above determined, so that we should have finally

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 4\pi\rho_1,$$

which is the analogue of the electrostatic equation connecting the total electric displacement with the free electrification.

6. Analytically, at a point inside a magnet there is no local term in the *actual* value of the potential of the magnetism; but the presence of relatively very great local elements in the summation which gives its value at any internal point prevents the force being derivable from that potential in the usual manner. That potential is in fact an actual physical instance of a function which is quantitatively continuous but not (practically) differentiable**: whereas the potential of the equivalent Poisson distribution of density is (sensibly) quantitatively equal to it at each point, and is also differentiable,—is in fact (thus far) an analytical function while the former is not.

But considerations of this kind are also logically necessary, even in the ordinary theory of gravitation. The gravitational potential of a system of molecules is a summation extended over the individual molecules, which is not an analytical function as regards its second differential coefficients; the value of one

** It appears that these terms are not here used in the rigorous mathematical sense. To avoid infinite values at poles, consider, for definite illustration, an assemblage of very small magnetically polarized spheres: when their number per unit volume is increased indefinitely and their size correspondingly diminished, the intensity of magnetization of each remaining the same, their magnetic potential in the spaces between them ultimately becomes an assignable continuous function. More precisely, its values along any line may be represented by the ordinates of a curve, which is a smooth curve on which is superposed an undulation of indefinitely small amplitude and wave-length, each of about the order of the distances between the spheres. In its gradient the amplitude of this zigzag that is thus superposed on the mean gradient curve will be finite, but its wave-length indefinitely small as before. Now we cannot, it is true, push on mathematically tc a limit, because molecular magnitudes are actually finite: yet if the unknown details of the molecular distribution cannot be eliminated, a mechanical theory of the properties of the medium is unattainable. What is done above is to define the function corresponding to the smooth curve as the analytical potential, and to define its gradient as the magnetic force. What is meant in the text is that in passing towards the limit the practically undeterminable minute local irregularities ultimately disappear from the potential, but become only more marked in its gradient. With the ideal continuity of an integral defined otherwise than as the limit of a process of this kind, and with the definite singularities which make a function non-analytical in the sense of exact Pure Mathematics, such as the precise but infinitely numerous oscillations of the function $x \sin x^{-1}$ in the neighbourhood of the origin, the principles of mathematical physics will possibly not be concerned; so that no confusion arises from the less definite use of terms in the text.

of these at a point inside the mass depends on the actual distribution of the molecules adjacent to the point. But as the result of the process of integration, actually performed or implied, we replace the actual potential of the molecular distribution by an entirely analytical function quantitatively equivalent to it, of which therefore the second differential coefficients are analytical: and it is this latter function V which is the real subject of the theory of the potential, and which is defined by the characteristic equation $d^2V/dx^2 + d^2V/dy^2 + d^2V/dz^2 = -4\pi\rho$, in which ρ is the averaged or smoothed out density of the molecular distribution. If on the other hand the distribution of density is entirely continuous, so that it has definite differential coefficients of all orders, integrations by parts performed in the usual manner of Green's theorem remain legitimate when the point attracted is inside the mass, so that the actual potential also has definite differential coefficients of all orders. The main problem of the transition from molecular to mechanical theory, which has presented itself several times in this book, is to determine the conditions under which an actual discrete or molecular distribution can be legitimately replaced by an analytically continuous one, that is the conditions under which the difference between them has no mechanical import.

Considerations of an analogous kind apply to the incompleteness of certain representations of a function by a Fourier series: the result is available quantitatively as regards the function itself and possibly its lower differential coefficients, but the equivalence does not extend to the higher ones. To turn this difficulty in physical problems, the method developed long ago by Sir George Stokes in his memoir on 'The Critical Values of Sums of Periodic Series,' namely to obtain an independent Fourier expansion of each differential coefficient, involving explicit terms in the coefficients arising from each place of discontinuity, must be adopted.

7. It is to be anticipated that considerations of convergency of the summations, similar to the above, will also arise with regard to the expressions for the electric and aethereal forces

dynamically induced in an element of volume situated within a magnetic medium. Starting with a linear molecular electric current to represent the ultimate discrete element of magnetism, as is required by the dynamical theory, the vector potential (F, G, H), which was determined (§ 55) to be $\int(u, v, w)\, r^{-1}\, d\tau$, will have an x component arising from this element of magnetism, of amount $i\int r^{-1}dx$ taken round the circuit of the molecular current i, which is by Stokes' theorem of transformation of integrals equal to $i\int\left(m\frac{d}{dz} - n\frac{d}{dy}\right) r^{-1}\, dS$, where dS is an element and (l, m, n) the direction vector of a flat barrier surface closing the circuit. Thus from the whole of the magnetism there arises in F the term

$$\Sigma i\int\frac{dx}{r}, \quad \text{equal to} \quad \int\left(\Sigma imdS\,.\frac{d}{dz}\frac{1}{r} - \Sigma indS\,.\frac{d}{dy}\frac{1}{r}\right),$$

that is,
$$\int\left(B\frac{d}{dz}\frac{1}{r} - C\frac{d}{dy}\frac{1}{r}\right)d\tau,$$

which is accordingly as regards outside points the x component of the vector potential of the magnetism. But at a point inside the magnetism r can vanish in an element of this integral: and the summation over the electrons which then represents the value of this component F, though itself representable quantitatively by this integral, is yet as regards its differential coefficients dependent on the unknown distribution of the adjacent electrons. This is of course quite in keeping with the fact that the magnetic field inside the magnet, considered as due to the aggregate of the molecular magnetic elements, by means of which the vector potential is defined through the relation curl $(F, G, H) = (\dot{\xi}, \dot{\eta}, \dot{\zeta})$ of § 55, itself involves this local distribution. But in the transition (§ 79) from a molecular to a mechanical theory, we have been able to discard the local part of the magnetic force, depending on the molecular character of the distribution at the point, from which alone indefiniteness arises. It may be surmised that we should in like manner discard from the vector potential the purely local contribution which is the source of its discontinuity.

This may be effected as usual by aid of integration by parts. At a point inside a minute cavity in the material medium, F, G, H and their first gradients can be represented analytically by integrals of the above type which are entirely convergent and determinate since r cannot be less than a finite lower limit in the integrals. On integration by parts each of these gradients is expressible as a volume integral together with an integral over interfaces of transition of the magnetism, and also an integral over the surface of the cavity: the volume integral is convergent and does not depend on the form of the cavity, while the integral over the surface of the cavity is finite and thus is the sole representative of the influence of the local molecular configuration; in our present procedure it depends on the form of the cavity; in actuality, when the cavity is merely the interstitial space between the surrounding molecules, it thus depends on the local molecular configuration. By the general principle, the mechanically effective functions are the analytical integrals obtained by excluding this undetermined local part. In the present case their values are found at once by transformation of the expressions for F, G, H, by integration by parts, so that the local contribution may be isolated and omitted. This leads to an expression for the total vector potential of the medium treated as continuous, of type

$$F=\int\frac{u}{r}\,d\tau+\int\left(\frac{dC}{dy}-\frac{dB}{dz}\right)\frac{1}{r}\,d\tau+\int(nB-mC)\,\frac{1}{r}\,dS,$$

the molecular currents which constitute the magnetism not being now capable of inclusion in the volume-distribution (u, v, w). It has already been verified in § 67 that this formula gives curl $(F, G, H)=(a, b, c)$. This vector potential is everywhere continuous. So is its gradient except at surfaces of transition of the magnetism, at which the component of the gradient along the normal is discontinuous on account of the surface integral: it is easily deduced that the circumstances of the transition at such a surface are expressible by continuity of the tangential component of the magnetic force (α, β, γ) and of the normal component of the magnetic induction (a, b, c): this is in fact the direct method of establishing those funda-

mental relations in the dynamical theory (Ch. VI) of a medium constituted of aether and ions and treated as continuous.

This determination of the value of curl (F, G, H) forms the analytical confirmation of the considerations in § 59 from which it was concluded that it is the magnetic induction that enters into the formula for the electric force. In that section it was explained that in estimating the mechanical force acting on the material medium, the part of the magnetic field which is of purely local origin should not be counted as contributing to the force on the electrons constituting the current in the element of volume, because its contribution will be cancelled by a reaction of the current on the magnetism in the element of volume. This mutual mechanical compensation of all internal actions in an element of volume is a corollary from the general principle of Action, which is taken to be fundamental and of universal application: it arises from the circumstance that the principle is expressible in terms of summation over the volume. Cf. also Appendix B, § 6. It is nevertheless desirable, for the sake of analytical precision and also for illustration of this principle, to point out where the reaction lies, in each case where it is possible to do so. In the present case the local mechanical reaction of the current in the element of volume on the magnetism in the same element is involved in the formula of § 65*. The magnetic field does not depend, as regards the numerical measure of its intensity, on the local distribution of the current: but its gradient, which enters into the force acting on the magnetism, does so, the integral expressing it in terms of the current not being convergent. In fact, the part of (α, β, γ) depending on the current is given by $\alpha' = dH'/dy - dG'/dz$ where $F' = \int \frac{u}{r}\, d\tau$, leading to $\alpha' = -\int \left(w \frac{d}{dy} - v \frac{d}{dz}\right) \frac{1}{r}\, d\tau$ in which the negative sign arises because the differentiation is now effected at the element of the integral instead of the point at which α' is estimated. Now this integral for α', or more strictly the summation over the individual discrete electrons for which it stands, is itself determinate irrespective of their local distribution, but its gradients, for

* On p. 104, line 6 from bottom of page should be $\gamma(v - \dot{g}) - \beta(w - \dot{h})$.

example $d\alpha'/dx$, are not so: each of them involves a part arising from purely local influence which will be discarded if we replace the expression for α' by the modified form $\alpha' = \int \left(\frac{dw}{dy} - \frac{dv}{dz}\right) \frac{1}{r} d\tau$ together with an integral over surfaces of discontinuity of the current. It is the part thus removed that is the reaction to the influence of the magnetism in the element on the current in the element which was itself excluded by the general considerations in § 59. Thus in the expression for X in § 65, the value of (α, β, γ) so far as it depends on the current must be that here given.

The general conclusion may be expressed, in an adaptation of Cauchy's terminology, by the principle that whenever the integrals in the formulae for mechanical forces on a material medium cease to be convergent, their principal values must be substituted ††.

8**. It has appeared (§ 70) as the result of an analysis in which each electron is separately accounted for as a singular point in the medium of propagation, namely in the free aether,

†† This statement may be considered to be the mathematical expression of the principle of the mutual compensation of molecular forcives, for which cf. *Phil. Trans.* 1897 A, p. 260. The principal value of Cauchy, as regards the completely defined analytical integrals of Pure Mathematics, would be the value at the centre of a minute spherical cavity. But the quantities which, to avoid periphrasis, have here been called integrals, are really summations of contributions from finite though very small, and complexly constituted, polarized molecules: the distribution of these molecules that occupy our minute cavity is entirely unknown and may be continually changing, so that the only possible principal value is the one that omits the contribution of neighbouring molecules altogether. It is an assumption that the function under consideration can be expressed as the sum of a purely local term and a definite part arising from the system as a whole; but in the special cases above considered this has been directly verified. If in any case it were not true, a dependence would be involved between mechanical change and molecular structure, so that mechanical causes would alter the constitution of the medium or even undermine its stability; whereas it is a postulate in ordinary mechanical theory that the physical properties of the medium are not affected by small forces. How far the impressed forces do actually thus operate can only be ascertained by actual trial in each case: even for the simpler cases in which the physical constants of the substance are definite functions of the strain, theory can hardly be more than a record of experimental fact.

that the term arising from the potential Ψ, which occurs in the expression for the electric force, is the statical force due to the instantaneous positions of the electrons. In fact an electron is a singular point in the aethereal displacement (f, g, h), such that over any region

$$\int (lf + mg + nh)\, dS = \Sigma e:$$

but
$$4\pi C^2 f = -\frac{dF}{dt} - \frac{d\Psi}{dx};$$

while
$$\int (lF + mG + nH)\, dS = 0$$

on account of the circuital property of (F, G, H); hence we have

$$\int \left(l\frac{d\Psi}{dx} + m\frac{d\Psi}{dy} + n\frac{d\Psi}{dz} \right) dS = -4\pi C^2 \Sigma e$$

as the equation determining Ψ. It follows from it that $\nabla^2\Psi$ vanishes everywhere in free aether, while Ψ in approaching an electron e becomes infinite of the form $4\pi C^2 e/r$: thus Ψ is the electrostatic potential of the distribution of electricity.

When the system is treated as a material medium in bulk, the effective density of the electric distribution is made up of ρ the density of the true charge of unpaired electric poles, and the density $-\left(\frac{df'}{dx} + \frac{dg'}{dy} + \frac{dh'}{dz}\right)$ representing the electric polarization. Thus

$$\int (lf + mg + nh)\, dS = \int \left(\rho - \frac{df'}{dx} - \frac{dg'}{dy} - \frac{dh'}{dz} \right) d\tau.$$

As all vectors such as (f, g, h) are now averaged into continuous functions as regards the medium in bulk, the left-hand side is equal to $\int \left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} \right) d\tau$: and since the equality of the two sides is maintained whatever be the region of integration, and also $f'' = f + f'$, we derive Maxwell's equation

$$\frac{df''}{dx} + \frac{dg''}{dy} + \frac{dh''}{dz} = \rho.$$

This relation, for a material medium not in rapid motion, is the same as

$$\frac{dKP}{dx}+\frac{dKQ}{dy}+\frac{dKR}{dz}=4\pi C^2\rho,$$

leading to

$$\frac{d}{dx}\left(K\frac{d\Psi}{dx}\right)+\frac{d}{dy}\left(K\frac{d\Psi}{dy}\right)+\frac{d}{dz}\left(K\frac{d\Psi}{dz}\right)$$
$$=-4\pi C^2\rho-\frac{d}{dt}\left(\frac{dKF}{dx}+\frac{dKG}{dy}+\frac{dKH}{dz}\right).$$

Thus in a heterogeneous dielectric the ordinary Maxwellian characteristic equation of this electric potential Ψ is not now satisfied. Nor is it satisfied at the interface between two homogeneous dielectrics, the surface density σ of true electrification being given by

$$4\pi\sigma=K_2N_2-K_1N_1$$

where N is the normal component of the force, so that

$$K_2\left(\frac{d\Psi}{dn}\right)_2-K_1\left(\frac{d\Psi}{dn}\right)_1=-4\pi\sigma-(K_2-K_1)\frac{d\mathfrak{N}}{dt}$$

where $\mathfrak{N}$ is the normal component of (F, G, H).

It has in fact been usual to conclude from equations such as these that in the Maxwellian electrodynamics Ψ is not the static potential of the electrification in the field. Whereas it here appears that it is merely the characteristic equation for Ψ in terms of the dielectric constants of the medium that has to be modified when the polarizing electric force is partly of electrodynamic origin.

APPENDIX B

ON THE SCOPE OF MECHANICAL EXPLANATION: AND ON THE IDEA OF FORCE

1. THE foundation of general mechanics, that is of the dynamics of material systems treated as continuous bodies instead of as molecular aggregates*, is formed by the following principles:

(i) The principle of equilibrium of mechanical forces, giving rise to ordinary statics, which may be summed up, in the manner initiated by Galileo and Newton, and ultimately fully developed by Lagrange, in the formula of virtual work:

(ii) The principle of d'Alembert, which asserts that the sensible motions of the mechanical system are an equivalent of the mechanical forces acting on it and in it; that is, if we set down the effective forces which would directly produce these motions in the separate parts or differential *elements of volume* of the system, considered by themselves as individually continuous but mutually disconnected, then for each part finite or infinitesimal of the system, these effective forces are the statical equivalent of the actual forces acting in or on that part either from a distance or through the adjacent parts.

It is convenient to designate a force acting on a part of the mechanical system from without as an *extraneous* or *impressed* force, and the forces arising from mutual stresses acting between two material differential elements of it (adjacent or not) as *internal* forces. It is explained in the science of statics that internal forces enter into the equation of virtual work only

* Cf. end of Appendix A; also Section II generally.

when the bodily configuration of the system undergoes change in making the virtual displacement to which that equation refers: herein lies the physical importance and fundamental character of that principle, and also its power in the application to systems partially unknown. Thus if we choose to consider a material system at rest as divided into two parts, then inasmuch as the mechanical forces acting on the whole system are in equilibrium, it follows that the forces exerted by the first part on the second equilibrate those exerted by the second part on the first. This argument applies not only to the complete material system, but to any portion of it that is *capable of separate continuous existence* independently of the contiguous portions: such a portion may itself be imagined as divided into two parts and the same conclusion drawn. Thus we have as a corollary to the first principle,

(i′) The mechanical action and reaction between any two parts of a material system, which are capable of separate permanent existence, must compensate each other, and therefore must have for their statical resultants equal and opposite wrenches (screw systems) on the same axial line.

This is the only form of expression of Newton's Third Law of Motion that admits of direct objective interpretation. The first and second of Newton's Laws of Motion are from our present standpoint descriptions of the course of mechanical phenomena which apply to the simpler cases. The scheme of laws which sufficed for the purposes of the *Principia* has in course of time gradually been broadened and developed by application to problems involving continuous bodies, of ever widening generality, until, mainly in Lagrange's hands, the science of Mechanics has again been reduced to a condensed and definite law of sequence in phenomena, in the form of the principle of Least Action. It is customary to deduce this general principle of Action from the simple Newtonian laws, by aid of an assumption that each of the material bodies is constituted of particles which act on each other with definite positional forces: but this by itself forms an extremely inadequate representation of the actual kinetic molecular structure of bodies.

2. The subject matter of the science of the statics of mechanical systems is thus definitely marked out: it consists partly in the mathematical verification of the equilibrium when the mechanical forces acting on and in the system, supposed of permanent configuration for the time being, are known; but more often it involves the mathematical determination of the existence and characteristics of forces of types not amenable to direct experiment, as derived from the conditions of their equilibration with other forces that are known. Inasmuch as the latter is the main aim, statics forms a branch of physical science, and is not wholly a mere geometrical calculus of forces.

In the general kinetics of mechanical systems the *rôle* is more complex. It is an easy matter,—or rather one free from any perplexity—to set down the effective forces that would be competent to produce directly the motions of the separate differential mechanical elements* of the system. These have to be equated to the actual forces acting on and in the system, of which the specification is a matter of greater nicety. Not to mention elastic stresses between strained parts of solid systems, whose theoretical development belongs to statics, there are cases in which the relative motions of the parts introduce passive reactions of frictional type whose nature and measure have to be elucidated as a preliminary to more refined applications of the dynamical theory. In most such cases it turns out that we cannot have a mechanical theory at all, except as a first approximation in those problems in which, the viscous forces being small, we can take them to be proportional to their originating circumstances; and a similar remark applies in fact also to statical elastic theory†.

Thus in the elementary development of mechanics, we begin

* It may be well to recall that by a mechanical element is meant the matter in an element of volume, considered as a permanent aggregate, without reference to its constituent molecules.

† It may not be superfluous to remark that a theory of transmission of stress in elastic matter exists in the ordinary sense, which considers the action of the surrounding medium on any given element of it as constituted solely of tractions over its surface, only because the range of the molecular forces of cohesion is very small compared with the dimensions of an element of volume which can be effectively treated as infinitesimal.

by forming the conception of a force, and formulating the principles that govern it in cases sufficiently simple to be amenable to exact observation and analysis: then by the application of these principles to more complex problems their scope is gradually widened, and it may be that they become less definite, while the conception of force is itself extended, until a stage arrives when a concise abstract restatement in more general terms of the principles of the whole subject is possible. All through this process, cases crop up in which the previously ascertained types of forces can only account for the observed motions by the help of new forces acting in definite ways, and these are henceforth to be reckoned with in causal connexion with the changes of configuration and motion taking place under all similar circumstances: thus there is continual addition being made to the types of forces which objectively exist. There are also other problems in which, when all known definite types are taken into account, additional forces are still needed in order to account for the phenomena, which we are unable to bring into any uniform causal relation except possibly of the roughest character with the bodily state of the system. In such cases the limits, for the time being, of exact mechanical science have been reached: an infinite intelligence that knew all about the constitution of the system (considered as made up of dead matter wholly controlled by physical law) might still be imagined as able to predict its future course, but to do so completely would probably require an amount of knowledge of its molecular details which transcends the limits of mechanics as here defined.

3. The foundation on which the whole subject is developed lies in the notion of forces. As mechanics took its origin in the equilibration of tendencies to motion of the various types that can be recognized, its chief concept lay to hand in muscular effort, which suggested a common standard of measurement. To make use of this concept for scientific purposes, precision in the method of measurement is, as usual, all that was required: theoretically a pull or a push can be measured to any degree of accuracy by the extension of a spring, or by use of the principle

of the lever, and the notion is thus ready for scientific development. To say, as is sometimes done, that force is a mere figment of the imagination which is useful to describe the motional changes that are going on around us in Nature, is to assume a scientific attitude that is appropriate for an intelligence that surveys the totality of things: but a finite intellect, engaged in spelling out the large-scale permanences of relations in material phenomena, is not cognizant of the bulk of these ultimate motions at all, and must supply the defect by the best apparatus of representation of the regular part of their effects that is in his power. When a person measures the steady pull of his arm by the extension of a spring, where or what, for example, are the motions of which the pull is only a mode of representation? The only way of gradually acquiring knowledge as to what they are, is to develope and make use of all the exact concepts that examination of the phenomena suggests to the mind. And in any case it is not the motions that are the essential factors, so much as the permanent entities of which the motions merely produce rearrangement.

Theoretical mechanics is thus an abstract science engaged in the application to natural phenomena of principles which are themselves in a state of development, mainly as regards detail, arising from the gradual reclamation of an empirical fringe surrounding the settled domain of the science. This fringe now extends a long way into molecular phenomena as distinct from mechanical, chiefly in the regions of the theory of gases and of radiant and electrical actions. Here progress has been effected mainly by transferring to the molecule, considered as itself a material system, dynamical ideas the same as or analogous to those that hold good in the mechanics of sensibly continuous bodies. But it is only the broad outlines of mechanical notions that can really be so applied, such for instance as Newton's laws were constructed to cover, involving only inertia and forces with reference to particles. And the reason why progress is possible at all is that the individual molecule is not an isolated thing, like one of Leibnitz's monads, jostling among its neighbours, but a nucleus in that universal aethereal *plenum* which is the transmitter of half our impressions, so that we can

learn about the phenomena of the individual molecule from the messages which are transmitted from the crowd of similar molecules to our senses through the aether.

This gradual development of mechanical principles is far from being terminated: yet no advance in method once gained is ever lost, though it will in time be transfigured into more perfect shape. There will always be an exact science of dynamics that will be rigorous within its own domain, in which domain alone—for the very reason that we are instructed by the science—we are able to take exact note of the more recondite uniformities of the complex of phenomena. But the greater part of these uniformities will always be beyond our ken: and it would not be legitimate to entertain any idea that, because our dynamical procedure has been an effectual means of mental coordination of physical events that are on a large and regular scale and of cognate types, it is therefore possible to lay down a finite scheme of principles by which the whole future course of inanimate material phenomena can be in any way reduced to rule.

3**. Nothing has been said here with regard to the frame with reference to which the motions of the parts of the system are to be specified. Philosophically we are accustomed to the standpoint that the motion of a body is unintelligible except in reference to some other body: yet in the formulation of dynamics the first thing that is done is to specify the motion of a body without any explicit reference to this other body, and this mode of procedure is not likely to be changed except perhaps superficially. If we adopt any aether theory, and so hold that material interactions take place in a *plenum*, it is unreasonable to suppose that this medium is disturbed except in the neighbourhood of the matter, and there is no occasion to try to change the dynamical procedure: for the *absolute* frame of reference to which motions are referred (*ultimate* frame is a better term, as it avoids the metaphysical suggestion) is the one determined by the distant undisturbed regions of the aether. A position of this kind does not in any way avoid a metaphysical or psychological analysis of the nature of space

and motion, but affords a reason why dynamical science is not required to pause until agreement on such questions is attained.

It is worthy of notice that in certain cases the method of reducing the number of coordinates, introduced by Routh and Kelvin, allows the application of the Action principle to the direct determination of the motion of a system relative to one of its constituent bodies. When we consider the motion relative to one of the bodies of the system, the additional coordinates entering into the Lagrangian function, which specify the actual motion of that body of reference, may be of the kind that can be ignored or eliminated by that procedure. Now the intrinsic internal forces of the system cannot depend on these coordinates, which represent its position in space but not its form: hence when they appear only through their velocities in the kinetic energy of the system, they can be eliminated. This condition is satisfied when the motion of the body of reference is one of translation, or is such a motion combined with rotation which is restricted to be about an axis fixed in direction*: thus it holds good for all two-dimensional problems: but when the rotational motion is unrestricted, its kinetic energy involves absolute angular coordinates as well as their velocities, and the independent determination of the relative motion is no longer possible. In the cases specified we can form a modified Lagrangian function $T' - W'$ for the system, involving only the relative coordinates, in terms of which the dynamical equations of the relative motion can be expressed in the form $\delta\int(T' - W')\,dt = 0$: in such a case T' and W' might be designated as the kinetic and potential energies of the system relative to the body considered.

But it is in all cases legitimate to refer the motion of the system to a frame, or set of axes, moving in any *prescribed* manner in space: for in the application of the Action principle the equations of constraint expressing the configuration in terms of the independent coordinates may involve the time explicitly.

* For particular examples, cf. Routh, 'Stability of Motion,' p. 67.

4. The science of Mechanics on its dynamical side thus treats of the relations of the motions of material bodies to the forces which produce them, while on the statical side it treats of relations of equivalence and of equilibrium between systems of forces. On the other hand it is not unusual to *define* a force in terms of the motion it produces in a material body: so that we become liable to the criticism that the development of relations between the forces and the motions of material systems is only a roundabout way of giving a mere description of the courses of the motions themselves. The subject of mechanical dynamics would then be reduced to the description of the sequence of the motions of material bodies: the notion of force would be a definition, and statics would be replaced by a theory of relations between mental concepts. On this view of Mechanics, which has always been more or less in evidence, but which is usually specially associated by continental writers with the name of Kirchhoff, the subject of the science is expressed in a more ultimate and ambitious manner as the formulation of the laws of the natural sequence of the motions of inanimate material systems: and the most scientific formulation of it would be one which deals solely with those motions and the simultaneous configurations of the system. Lagrange, in one of his earliest and most important memoirs*, had already established an ideally perfect foundation for the science from this point of view, in his extension of the principle of Least Action to general mechanical systems. That principle in one of its forms asserts that, however complex the system may be,—provided it is self-contained and conservative in the sense that the forces depend only on its actual configuration and therefore, by the fundamental induction of the conservation of energy, are derived from a potential energy function—the natural course of its motion from a configuration A to a configuration B is one which makes the time-integral of the difference between its kinetic and potential energies stationary as regards slight variations of the path of the system between these configura-

* "Essai d'une Nouvelle Méthode pour déterminer les *maxima* et les *minima* des formules intégrales indéfinies," "Application de la méthode précédente à la solution de différens problèmes de Dynamique." *Mémoires de Turin*, 1760–61.

tions, the time of passage being supposed unvaried; if the configurations A and B are sufficiently near each other in time, this time-integral, first explicitly defined as the Principal Function by Hamilton, is *minimum* and the least possible, and the course of the natural motion is unique; if they are further apart it need not be *minimum* †, the change being associated with the circumstance that there may then be various slightly different courses of natural motion between the configurations.

This principle then comprehends in itself the whole of mechanical science. For example, it involves the formula of kinetic energy in the form that the sum of the kinetic and potential energies remains constant throughout the duration of the motion: that is however a different thing from asserting that it includes the principle of the Conservation of Energy or complete negation of perpetual motions, which is already previously involved in the assertion of the existence of a potential energy function**. The statement of the principle may however be extended so as to be independent of this restriction to conservative systems, by including in the equation of variation the work of any other impressed forces, either extraneous or internal, that are not already involved in the potential energy; it is then the variation of the Principal Function for any virtual displacement of the course of motion,

† It can be *maximum* only with regard to a restricted variation, the simplest case being that in which there is a number of 'kinetic foci' conjugate to A, along AB, which is at least one less than the number of degrees of freedom of the system. Then if C is an intermediate configuration, to which there is no focus conjugate along CB, the Action along AB is greater than that along AC_1 and C_1B, where these are two natural paths uniting at any configuration C_1 very near to C. Thomson and Tait, *Nat. Phil.* § 364. In the case of a system of optical rays diverging from A, and ultimately straight between C and B, the sufficient (and equivalent) geometrical condition is that the wave-front at C, belonging to a wave-train from A, shall be wholly convex towards the side B of its normal CB.

** This restriction really means that the principle of Action, in its simple form, is applicable only to a system whose mechanical constitution is permanent so that its sequence of configurations is expressible by purely analytical functions, and which therefore can be restored to any previous state by purely mechanical means such as reversal of its sensible velocities.

together with the virtual work of these additional forces in that displacement, that is to vanish. In this form the application of the principle is universal, subject to the one restriction that the configuration of the system is always completely expressible in terms of the system of coordinates that is employed, without requiring the introduction of their time-fluxions, and that these coordinates are themselves independent: but on the other hand the idea of force has been imported into it. The restriction on the coordinates, just mentioned, is a very important one, as it excludes from the principle most rolling motions of rigid bodies: one of the relations by which the number of coordinates would be in such a case reduced to that just sufficient to specify the configuration, without the redundance which would be fatal to their mathematical independence, is that of equality of the tangential velocities of the two rolling bodies at their point of contact, and this relation would involve in its expression the velocities or time-fluxions aforesaid. In such cases we must avoid restricting the motion by relations of this sort; it is therefore necessary to introduce new coordinates equal in number to the relations so ignored; but we must then introduce into the Action formula, along with the other extraneous forces if any, undetermined parameters representing the equal and opposite forcives which act between the rolling bodies at their places of contact, or in connexion with whatever additional modes of freedom are thus contemplated. The vanishing of the variation in this more general form, when the virtual displacement of the motion is extended so as to include sliding, will then determine the general course of the system, the tangential force being determined at the end of the process so as to satisfy the kinematic condition of absence of sliding*.

* Hertz, in his posthumous 'Principien der Mechanik,' saw in this exclusion of pure rolling a reason for refusing to base general mechanics on the Action principle, as being devoid of the requisite generality: and his book is chiefly occupied with the quest of another principle to take its place. Against this view it may be urged that the notion of rolling is foreign to molecular dynamics, on which the laws of mechanical dynamics must be ultimately based. The criticism has also been made that the principle of Least Action cannot be philosophically fundamental, inasmuch as it determines the present course of the system by a reference to its future as well as to its past: this objection will

5. This sketch indicates how far it is possible to eliminate force as an independent concept from the treatment of mechanics: it shows that we can do so only in the case of non-dissipative systems, and then only when there are no rolling motions. But, on the other hand, it would be a misfortune to banish the idea of force even if we could. Any definition that would merely make it a subjective cause of motion is incomplete: the concept is required for the expression of properties of permanent groupings of natural phenomena, and in that sense is as much objective as anything else. When we hang up a given weight on a spring balance the extension of the balance is always the same, subject to permanence of locality and other assignable conditions, and whenever we see the spring so extended we infer at once that it is supporting an equal weight or else doing something equivalent: we say that it is exerting a certain definite force. It would of course be useless to introduce this conception of force if the uniformity of the course of Nature did not hold to the extent here described. But as it does hold, the force is the concept that allows us to eliminate the consideration of the complex of changes of molecular states and motions that is involved in the extension of the spring, of which we know nothing except that they are for our purposes the same in each case. In the mechanics of permanent material bodies the idea of force is thus essential in order to avoid wholly unknown molecular considerations: it results from and is the sufficient expression of dynamical permanence and the extent to which it is known from experience to exist in similar cases: in the absence of this concept there would exist dynamics of molecular systems, but there could be no mechanics of finite bodies. In a purely analytical formulation, force would now be defined as a coefficient in the variation of the mechanical part of the Action. In so far as this point of view is admitted, it will

be removed if we bear in mind that the complete system is of very complex molecular constitution, and that the principle of Action is really only an algorithm constructed so as to enable us to abstract the molecular details while retaining all that relates to the matter in bulk. A similar remark applies to the principle of virtual work, which is included as a special case in the wider principle of Action. Cf. §§ 6—8 *infra*.

follow that the notion of any special 'relativity of force' is the result of a misapprehension.

On the other hand, if the idea of force had not been supplied to us ready formed, through our muscular sense, we can conceive that the science of Mechanics must have begun with the dynamics of molecular systems, and the forces between permanent finite bodies would have been discovered and defined as new physical conceptions simplifying the theoretical discussions and related to the degree of permanence of the systems: the conception of potential in electrostatics is actually one of this kind: so is that of temperature, which also was early developed because our sense of heat supplied it ready formed. There is a whole series of such conceptions, derived in part from theory and in part from experiment, on which the structure of electrical science rests: moreover these are not to be resolved into ultimate elements by the easy process of talking about molecules and the forces acting between them: for they involve the aether as well as the molecules, which perhaps would not matter but for the fact that they involve it in such a way that it makes an important difference to them whether the matter is at rest or in rapid motion through the aether. The ideal logical method of developing them would be to begin with the complete system of aether *plus* discrete molecules, and afterwards deduce the concepts and laws which apply to the mechanical system of aether *plus* matter specified by its properties in bulk. It is only however by the process of trial and error, in conjunction with generalizations derived from the experimental scrutiny of matter, that we can safely learn to include in this discussion the part of molecular relations that is essential and permanent for the field of phenomena in view, and to omit the other part that does not bear on them: and it is in this way, under the constant guidance of observation and experiment, that the dynamical side of abstract physical theory advances.

6**. *Molecular Basis of General Dynamics.*—When once it is allowed that the seat of the activity, in dynamical phenomena, is the pervading aether, it readily follows, from the equation of Action $\delta\int(T-W)dt=0$ which determines the sequence

of states of that medium, that $T+W$ is equal to E, a constant as regards time for any region of it not under external influences, which is called the energy. By analytical transformation this aethereal energy may be expressed as in the main attached to the molecules of the material system: and when it is finally transformed partly into a mechanical specification depending on the configuration and motion of the matter in bulk, partly into an irregular residue of uncoordinated molecular motions or heat, and partly into internal or chemical and radiant energy of these molecules individually, we arrive at a *rationale* of the principle of the conservation of total energy such as has been formulated as a result of universal experience.

In preparing this general Action equation, thus supposed given in its exact and fundamental molecular form, for mechanical applications, it is obviously incumbent to introduce all the coordinates of the system, regarded as matter in bulk, that are available; when this has been completely accomplished the remaining independent translational coordinates belonging to the individual atoms will be of rapidly fluctuating sign. The variation conducted with regard to the former mechanical coordinates should now lead to dynamical equations for the mechanical system. In these equations each molecular coordinate, wherever it appears, may be replaced by its mean value zero, as also may its velocity, and acceleration, while the squares and products of such quantities may be replaced by mean values. But this process ought, in so far as a mechanical analysis is possible, to be performable* equally well in the

* This will involve a restriction on the form of T. If ϕ represent a mechanical coordinate and ψ a residual molecular one, the types of terms that can come into T are included in

$$[\phi\psi]\,\dot\phi^2,\quad [\phi\psi]\,\dot\psi^2,\quad [\phi\psi]\,\dot\phi\dot\psi,$$

where $[\phi\psi]$ represents a mixed function of the two kinds of coordinates. For the interchange of order of these operations to be possible, namely of the Lagrangian operation $\frac{\delta}{dt}\frac{d}{d\dot\phi}-\frac{d}{d\phi}$ and the process of averaging, it is necessary that the third type of term should be restricted to the form $[\phi\phi]\,\dot\phi\dot\psi$. Under the circumstances of Appendix F, the kinetic energy of a molecule is of type $\frac{1}{2}\Sigma m\,(\dot x^2+\dot y^2+\dot z^2)$ where m is a constant for each primordial atom: the energy will then be transformable into the type thus restricted, and the condition will be satisfied.

formula for the Action before the variation takes place; and the result of it will then be the expression for the mechanical Lagrangian function of the system $T' - W'$, from which the mechanical energy-function E' may be derived by a change of sign of the terms not involving velocities. The variation of the Action with regard to the individual purely molecular coordinates would not in any case usually lead to results lying within the range of experience, so that our want of knowledge of the form of the Lagrangian function in this respect is not material. But the mode of execution of the mechanical variation here described assumes that the mean squares of the molecular coordinates and velocities, for the smallest time that is sensible, are steady throughout the motion, or throughout that part of it which is for the purpose in hand treated by itself; this implies that the system is not undergoing constitutive change and that it is in a steady thermal state: the effect of change in either of these respects is to produce continual alteration of the coefficients in the mechanical energy-function*.

This analytical formulation of mechanical dynamics is therefore an ideal limit, applicable to systems which are molecularly steady, or conservative, for the kind of motions under consideration, so that the system can always theoretically be restored to any previous state by mechanical means alone: in other systems the separation of a mechanical part of the Action is not possible, or is a sensibly imperfect process which may be empirically amended by the introduction of new forces, of frictional or other irreversible type, suggested by observation and experiment. In other words, there can be no question of demonstrating the principle of Action for any existing system, but rather of examining the course of a limited number of particular dynamical processes in such a system in order to form a judgment as to whether, as regards the totality of its mechanical relations, it may be included with sufficient approximation in this ideal conservative type†.

* Cf. Liouville's problem of the dynamics of a solid body which is contracting owing to loss of heat.

† The manner in which the application of this mechanical principle, once admitted, is to be conducted, cannot even now be more perfectly expressed than

7**. *Molecular Basis of Thermodynamics.*—The Lagrangian dynamics of mechanical systems is thus involved in its entirety in the molecular foundation from which we have started, namely, the dynamical equations of the free aether combined with the present conception of molecules, provided the molecular state of the system is steady: but the principle of Action can effect nothing for us in the matter of the sequence of individual molecular changes because the number of independent molecular coordinates transcends all calculation or even conception. The foundation of thermodynamics must thus be formulated in some other way: and this will be made easier if we first realize, in concise abstract statement, what are the principles in that subject which are to be explained.

Beyond any doubt, the fundamental thermodynamic relation is the law of equality of temperatures, which asserts that when a heterogeneous material system is in a steady condition there is a function of the physical state, definite and assignable for each kind of matter and each element of the system, called the temperature, which has the same value throughout it. When this is not constant throughout there will be molecular changes of a non-oscillatory character, involving transference of energy from a part of the system in which the temperature is higher to adjacent parts in which it is lower. This finite transference of energy, which is the one tangible result as regards matter in

in Green's original discussion in the introduction to his memoir 'On the Reflexion and Refraction of Light,' *Camb. Trans.* Dec. 1837. The fundamental analysis by Sir George Stokes 'On the theories of the Internal Friction of Fluids in Motion, and of the Equilibrium and Motion of Elastic Solids,' *Camb. Trans.* April 1845, does not detract from the present argument: the main question there is as to the possible physical grounds for the relation imposed by Poisson on the two elastic constants in Green's potential energy formula: it would appear that any settling of the molecules in a strained body towards a new configuration, such as that there contemplated in § 19, must involve some orderly process in so far as it can play a part towards determining the values of the elastic constants which belong to the element of volume of the material. There is however one way in which the uncoordinated local molecular motions can affect the physical constants of the material, namely, through their aggregate which determines the temperature: an example is the effective elasticity of the air for sound waves of rapid period: to obtain the complete formulation for such cases thermodynamic considerations must be included as in the following section.

bulk in a permanent state, is known as transfer of heat, and heat is in fact objectively measured directly as energy. What has here been called the law of equality of temperatures asserts that if there are three bodies A, B, C, so that A is in contact with B along an interface and also B with C, each in thermal equilibrium, then when B is removed and A and C moved into contact without alteration of their molecular states, they will remain in thermal equilibrium. A reason must be assignable for this law, which is by no means formally necessary. It follows however from Maxwell's fundamental remarks on the possibility of establishing differences of temperature merely by the aid of constraint applied to the individual molecules, that a purely dynamical explanation is not possible, that it is really a question involving the average state of a large number of molecules. Suppose now that our three bodies A, B, C are enclosed in an adiabatic envelope: if the law of equality of temperature do not hold, we can by a series of alterations of their mutual contacts (which may be imagined as set going permanently by an automatic arrangement) cause a redistribution of their internal molecular energy at each operation, and by suitable mechanical arrangement we can during each redistribution guide some of it off to be added to the mechanical stock of energy outside the envelope. These mechanical operations may be conceived to go on, involving gradual transformation of the molecular energy of the system into mechanical energy outside, until possibly a stage arrives at which the law of equality of temperatures holds good inside the enclosure: and then the process stops. But we have only to enlarge our system by connecting it thermally with some other system, to start the process again. Thus unless the law of equality of temperature in steady states holds, it will be theoretically possible by automatic mechanical arrangements and with systems in a steady state, to convert a finite fraction of the molecular energy within our reach into mechanical energy. The negation of this possibility carries with it the law under consideration: and this negation can only rest broadly on the discrete character of matter which makes the frittering away of mechanical motions into irregular molecular ones a natural process, but effectually prohibits on any realizable scale

the reverse process. The law of temperature then stands on this basis alone. There are it is true cases in which alteration of the contacts does involve redistribution of the molecular energy, those namely in which the new contacts allow chemical or constitutive change to begin: such change must be excluded in the statement of the thermal proposition, which relates only to the irregular translational and rotational energy of the molecules that is not bound up with their chemical constitution†. The law should thus rather be stated in the less definite form that wherever a difference of temperature is maintained between two portions of matter in contact with each other, a special cause for it must be assignable: and this will be in keeping with its origin as a physical conception rather than a dynamical principle.

As regards the physical nature of this quantity temperature, it is clearly related to the *squares* of the molecular velocities, which is another reason why it cannot have any direct connexion with the dynamical coordinates of the system.

If the foundation of the law of temperature on this basis is allowed, the same mode of argument will carry us much further. Consider the system as before in its adiabatic enclosure: if physical or constitutive changes of state in it are possible, they will not occur when their result would be to increase the mechanically available part of the energy of the system*: the states of mere equilibrium of the system, that is of infinite slowness of incipient change, are therefore the ones in which the available energy is stationary as regards all infinitesimal changes: the states of stability are those for which the available energy is an actual minimum. The available energy belonging to a definite state of a given piece of matter is a function of that state alone, determined by the amount of mechanical work that can be, theoretically, gained in the transition in a mechanically reversible manner to some standard state of the same portion of matter. In the state-

† An enclosed region of free aether might be taken as one of the bodies of the system, its temperature being defined in terms of the (extremely small) density of the vibrational energy that pervades it.

* Rayleigh, 'On the Dissipation of Energy,' *Proc. Royal Institution*, 1875: 'Collected Papers,' i, p. 240. Cf. also Rankine, 'Scientific Papers,' p. 311, 1853.

ment of the principle we may get rid of the adiabatic enclosure, and simply say that for stability of the system the mechanically available part of the energy must be a minimum for given amount of total energy. This available energy is the characteristic function of Willard Gibbs: and the statement here made is the most complete as well as direct form of the second principle of thermodynamics. The main business of that science is the gradual determination, by experimental aid, of expressions for the amounts of available energy inherent in different kinds and states of matter: this is a process which will still be progressive, as new modes of availability are continually recognized, and in which rigorous finality is not to be expected: for instance, if an apparent violation of this second principle of thermodynamics is observed, through apparent spontaneous increase of availability in one direction, search must be made for an unrecognized availability of some other kind which is diminished by at least an equal amount*.

In this view of the origins of thermodynamic doctrine, no special *rôle* has been assigned to steady gyrostatic motions such as could be eliminated from analytical dynamical theory by the process of Routh and Lord Kelvin. The invocation of concealed steady motion, or gyrostatic relations, towards the dynamical explanation of physical phenomena, has been prominent all through Lord Kelvin's writings, a notable instance being that of magnetic optical rotation, which so strongly impressed Maxwell's mind. In this and other cases (*e.g.* the explanation of elasticity) it was always a definite dynamical action that was to be accounted for. But more recently von Helmholtz has made a sustained attempt to elucidate the dynamical laws of heat by making heat analogous to the energy of concealed motions, treated on this basis of a modified Lagrangian function. His analogies in this direction obtained an amount of success which has been variously estimated. They represent the standpoint of ideal molecules with gyrostatic quality, linked with each other through a finite number of mechanical connexions rather than influencing each other

* For further development on these lines cf. *Phil. Trans.* 1897 A, pp. 260 sqq.

through the aethereal medium: and for such a definite system they would explain the thermodynamics of reversible processes in which there is absorption but not dissipation of energy.

8**. One effect of admitting a molecular synthesis of dynamical principles such as the one here described is to depose the conception of energy from the fundamental or absolute status that is sometimes assigned to it; if a molecular constitution of matter is fundamental, energy cannot also be so. It has appeared that we can know nothing about the aggregate or total energy of the molecules of a material system, except that its numerical value is diminished in a definite manner when the system does mechanical work or loses heat. The definite amount of energy that plays so prominent a part in mechanical and physical theory is really the mechanically available energy, which is separated out from the aggregate energy by a mathematical process of averaging, in the course of the transition from the definite molecular system to the material system considered as aggregated matter in bulk. This energy is definite, but is not, like matter itself, an entity that is conserved in unchanging amount: it merely possesses the statistical, yet practically exact, property, based on the partly uncoordinated character of molecular aggregation, that it cannot spontaneously increase, while it may and usually does diminish, in the course of gradual physical changes.

A simple example of this separation of a mechanical portion of the energy is furnished by the phenomena of osmotic pressure. In a solution each molecule of the dissolved substance is the centre of an aggregate of those molecules of the solvent which are within its range of molecular action and so are to some extent affected by it. In a concentrated solution these aggregates run into each other; but when the dilution is great, they are independent systems separated by the unaltered solvent. There is a part of the mechanical energy which arises from the mutual presence of the two kinds of molecules: this part can depend, per unit volume, only on the number of the molecules of the dissolved substance, the temperature, and possibly the kinds of matter involved. But when the solution is very dilute,

further dilution by addition of more of the solvent will merely separate these molecular aggregates in space without interfering with the constitution of any of them: thus the *change* of the mechanical energy which arises from further dilution is then independent of the kind of matter involved, and can depend only on the number of molecules of the dissolved substance per unit volume, being the same for all systems which agree in this respect. One such system is an ideal gas, in which the molecules are supposed to be, for practically all the time, outside each others' sphere of influence, there being now no solvent: it follows that the osmotic pressure of the molecules of the dissolved substance against an internal partition, permeable to the solvent but not to them, is the same as if they existed in the state of an ideal gas at their actual density and temperature*. It can be urged that these considerations amount to demonstration rather than explanation, that they compel assent rather than satisfy the mind: indeed the nature of the validity of this most remarkable generalization was in doubt for years after experimental facts had compelled its recognition by van 't Hoff. But in fact similar considerations enter in forming the mechanical energy function of any material system: if the system is not dissipative, i.e. if it does not gradually run down in the course of mechanical transformations, it must have a mechanical energy function: the form of this function cannot be derived from its molecular constitution, of which we shall possibly never obtain sufficient knowledge for such an application, but from indirect reasoning guided by observed mechanical properties of the system. Thus practically, in Newton's words, the whole problem of Natural Philosophy is concerned in this, 'ut a phaenomenis motuum investigemus vires naturae, deinde ab his viribus demonstremus phaenomena reliqua.'

9**. It would appear (p. 260) that there can be an unlimited amount of molecular structure and function in a given system, which is unconnected with any mechanical effect occurring in that system treated as continuous matter. This is because, whether we view it as an independent principle

* Cf. *Proc. Camb. Phil. Soc.* Jan. 1897: *Phil. Trans.* 1897 A, p. 275.

or as a corollary from the doctrine of Action, these mechanical relations are from their very nature determined by analytical functions of configuration and their first and second gradients alone: if higher gradients also came in, the statement would no longer hold. The processes by which our conception of the uniformity of Nature is obtained essentially involve averaging of effects, and lose their efficacy long before the individual molecule is reached. Mechanical determinateness thus need not involve molecular determinateness: then why should either of them involve determination in the entirely distinct province of vital activity?

Moreover mechanical science has to do with systems in being: it does not avail to trace the circumstances of growth or structural change even in inorganic material. What happens when two gaseous molecules unite to form a compound molecule is unknown except from the slight indirect indications of spectrum analysis. Now all initiation of organic activity seems to involve structural change, not merely mechanical disturbance, and is, in so far, outside the domain of mechanical laws. But the activities of an organism treated as a permanent system—such for example as propagation of nervous impulse—are likely enough, when once they are started, to be of the nature of the interactions of matter in bulk, so that it is legitimate to seek for them a mechanical correlation. Every vital process may conceivably thus be correlated with a mechanical process, as to its progress, just to that extent to which it is possible experimentally to follow it, without lending any countenance to a theory that would place its initiation under the control of any such system of mechanical relations. In other terms, there is room for complete mechanical coordination of all the functions of an organism, treated as an existing material system, without requiring any admission that similar principles are supreme in the more remote and infinitely complex phenomena concerned in growth and decay of structure.

APPENDIX C

ON ELECTROLYSIS: AND THE MOLECULAR CHARACTER OF ELECTRIC CONDUCTION

1. THE fundamental facts to which a theory of electrolysis must conform are as follows:

(i) Faraday's law; that the number of molecules of the anion liberated in any time is the same as the number of molecules of the cation, and that corresponding to the liberation of one molecule of either of them the same quantity of electricity passes in the current, a quantity which on comparing different anions and cations is proportional to their chemical valencies; a definite quantity of electricity—the fundamental unit of charge or the electron—thus corresponding to each valency, whatever be the electrolytic substance:

(ii) Kohlrausch's law that the conductivity of a very dilute electrolytic solution is (for a given solvent) the sum of two parts, one characteristic of the anion alone and proportional to the strength in which it is present, the other similarly characteristic of the cation alone.

The second of these facts has suggested the view that in a dilute electrolytic solution the anion and the cation are effectively independent of each other as regards mobility, while each carries as an electric charge the number of electrons represented by its valency. Now in each element of volume the numbers of anions and cations must be the same to an extremely close approximation, because—the electron being an enormous charge relative to the mass of the molecule—a very slight discrepancy between the numbers of positive and negative ions would imply a large volume density of electrification. The view would then be that the electric force

(electromotive force per unit length) in the solution urges these ions, in virtue of their charges, in opposite directions and thus establishes steady drifting motions of the two ions which have different velocities as Kohlrausch's law indicates. But here we are in danger of coming into collision with Faraday's law; for the supply of molecules of that ion which has the greater mobility would be in excess at its electrode, whereas the numbers of molecules liberated at the two electrodes are really equal. There must thus tend to be an accumulation of these ions in excess, around their electrode, that will have to be somehow relieved, and the electrolytic current will not remain a steady phenomenon: the extent of this unsteadiness it is important to ascertain.

At the very beginning of the conduction in a fresh uniform solution, let the averaged velocity of drift of the cation, in accordance with the hypothesis, be V_1 say to the right, and that of the anion V_2 to the left. Let us decompose this velocity of the cation into $\frac{1}{2}(V_1+V_2)$ to the right and $\frac{1}{2}(V_1-V_2)$ to the right: and in the same way let us decompose the velocity of the anion into $\frac{1}{2}(V_2+V_1)$ to the left and $\frac{1}{2}(V_2-V_1)$ to the left. On pairing these components we shall have a drift of the two ions right and left with equal speeds each $\frac{1}{2}(V_1+V_2)$, and a drift of them together in company to the right with speed $\frac{1}{2}(V_1-V_2)$. The former represents a current of conduction obeying Kohlrausch's law, and involving no accumulation of ions at either electrode: the latter represents a uniform flow of the electrolyte itself without any electric separation, and leads to an increase of density in the solution up against that electrode—say the cathode—whose ion has the greater velocity of drift, with a corresponding decrease of density up against the other electrode. This piling up of the electrolyte is partly relieved by ordinary diffusion back again; and the initial aggregate changes of concentration in the neighbourhood of the electrodes form the well-known phenomenon investigated by Hittorf*.

* *Pogg. Ann.* 1853—8: cf. Winckelmann's 'Physik' III, i, p. 449. In the initial stages the gradient of concentration near the middle is negligible, hence V_1 and V_2 are there proportional to the mobilities of the ions: and whatever be

2. The question before us is how far this state of affairs can be regarded as permanent; or whether changes will supervene in the process in the course of time which will among other things involve alteration in the mode in which the solution conducts the current. The heaping up of the electrolyte towards the cathode and away from the anode will go on, at diminishing speed, until the steady stage arises when diffusion backwards through the liquid just balances the drift forward under electric force. And here we must decide between two hypotheses.

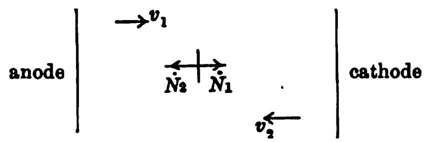
(i) We might assume that the electrolyte diffuses back as a sugar solution would do, without separation of the ions: in that case the nature of the electric conduction would not be affected, and we might calculate the conductivity of the solution, of varying density between the electrodes, by the same rule as applies to a wire of varying section.

(ii) We might assume, as is much more in keeping with the hypothesis, that the ions of the electrolyte have independent mobilities as regards diffusion just as they have as regards drift under electric forces: their different speeds of diffusion backwards will now initiate electric separation and consequent bodily electrification in the solution, which will react so as to affect the electric transfer and may possibly in time fundamentally alter the nature of the conduction.

In attempting to trace what will happen on the second hypothesis, it will be a great simplification to assume that the numbers of positive and negative free ions are always the same, say n per unit volume, in each part of the solution: this will be practically true because n is very large, so that an excessively small relative difference in the numbers of positive and negative ions would imply a very great electrification. It is equivalent to assuming that the electric current I is precisely the same across all sections of the electrolyte at each instant of time.

the shape of the cell, the mass transported across the middle is to the mass electrolyzed in the ratio $(V_1 - V_2)/(V_1 + V_2)$: thus the total amount of the cation that is transported is to the amount of it that is electrolyzed in the ratio $V_1/\frac{1}{2}(V_1 + V_2)$, which is Hittorf's transport number for that ion.

We assume for the sake of simplicity that the solution is so dilute that it is completely ionized.



Let us fix our attention on the cross section at distance x from the anode, and suppose that dN_1/dt cations cross it in one direction (along x increasing) and dN_2/dt anions in the opposite direction per unit area per unit time. These movements are due in part to the electric force $-dV/dx$ at the place, and in part to diffusion: thus

$$\frac{dN_1}{dt} = -nv_1\frac{dV}{dx} - k_1\frac{dn}{dx};$$

here v_1 is the velocity of drift of the cation under unit electric force, as determined indirectly by Kohlrausch and first visually exhibited by the experiments of Lodge; and k_1 is a constant independent of n, which is a coefficient of diffusion of these ions of the ordinary type. Similarly

$$\frac{dN_2}{dt} = -nv_2\frac{dV}{dx} + k_2\frac{dn}{dx}.$$

The continuity of electric flow gives

$$\frac{dN_1}{dt} + \frac{dN_2}{dt} = \frac{I}{e}$$

which is constant along the flow, e being the ionic charge, positive for the cation negative for the anion, and I the electric current. The continuity of flow of the electrolyte gives

$$\frac{d}{dx}\tfrac{1}{2}\left(\frac{dN_1}{dt} - \frac{dN_2}{dt}\right) = -\frac{dn}{dt}.$$

These equations represent a complete scheme of the course of the phenomena: thus, for example, there are four equations involving four independent variables N_1, N_2, V, n when the current I is maintained constant, or again the electromotive force $-\int dV/dx\,.\,dx$ may be maintained constant, when I will vary with the time.

3. We can assign theoretically the values of k_1 and k_2, if, after Nernst, we follow out the hypothesis of effectively independent mobility of the anion and cation into its natural consequences in the domain of osmotic theory. For the sake of precision the osmotic argument will be indicated in full. Consider an ordinary solution, say of sugar, separated from a mass of the pure solvent by a diaphragm: the phenomena of osmosis suggest and warrant the theoretical statement, that if a diaphragm is postulated of a kind that is freely permeable to the solvent but wholly impermeable to the molecules of the dissolved substance, then the pure solvent will creep through it until there is a definite difference of fluid pressure established between the two sides. Experiment has suggested and verified the law that when the solution is dilute, so that the dissolved molecules are at distances apart comparable on the average to those of the molecules of a free gas, this osmotic pressure is the same as, and may be represented by, the pressure of these dissolved molecules against the diaphragm, considered as if they constituted a free gas with their actual distribution and temperature. This principle admits of rigorous thermodynamic proof, which is immediate from the point of view of available energy*. It forms only another way of expressing this law, to say that the osmotic pressure is the force per unit area, or the 'partial pressure,' that must be applied (by the diaphragm or otherwise) against the dissolved molecules in bulk, considered by themselves, in order to prevent their diffusion. It follows again, by way of reaction, that in case of a solution of varying concentration, the force operating to cause translation, by diffusion, of the molecules in a thin slice of the solution, is the reversed difference of the osmotic pressures on the two faces of the slice. Now in the case of either set of ions we know by experiment the velocity of drift produced by an ascertained applied bodily force of electric type, and the same coefficient of drift will naturally apply when the sifting force is of osmotic type: thus k_1 and k_2 are known in terms of v_1 and v_2. In fact the osmotic law then gives for the pressure the formula

$$p = neRT,$$

* Cf. *Phil. Trans.* 1897 A, p. 272: or p. 287 *supra*.

where R is the constant of perfect gases and is the same for all kinds of ions, and T is the absolute temperature: also the effect of the osmotic force $-dp/dx$ compares directly with that of the electric force $-ned V/dx$ which produces the drift $-nv_1 d V/dx$, hence the osmotic drift is $-\frac{v_1}{e}\frac{dp}{dx}$ or $-v_1 RT\frac{dn}{dx}$ so that

$$k_1 = v_1 RT, \qquad k_2 = v_2 RT.$$

4. But for the present we shall retain k_1 and k_2 as independent constants, and thereby postpone the assumption that the anion and cation are permanently dissociated in the dilute solution. We might in fact, thus far, base the equations on the original theory of mobile association (so to speak) of Williamson and Clausius, assuming simply that when the anion and cation of a molecule happen to get knocked asunder, the greater mobility of one of them carries it further than the other, under the influence of electric force or of diffusion, before fresh partners are acquired. On either view we must have $k_1/v_1 = k_2/v_2, = \beta$ say; it is only the special value RT for β that the theory of complete dissociation supplies.

Towards solving this scheme of equations we have, since dI/dx is null,

$$\frac{d^2N_1}{dx\,dt} = -\frac{d^2N_2}{dx\,dt} = -\frac{dn}{dt}:$$

hence

$$-\frac{dn}{dt} = -v_1\frac{d}{dx}\left(n\frac{dV}{dx}\right) - k_1\frac{d^2n}{dx^2}$$

$$\frac{dn}{dt} = -v_2\frac{d}{dx}\left(n\frac{dV}{dx}\right) + k_2\frac{d^2n}{dx^2}.$$

These are the differential equations determining the course of the distribution of density of the electrolyte, specified by the variable n, and of the distribution of electric potential, as time proceeds.

They give immediately for the former by itself

$$\frac{dn}{dt} = \frac{k_2v_1 + k_1v_2}{v_1 + v_2}\frac{d^2n}{dx^2}.$$

But this is precisely the type of equation of diffusion that

would hold for an ordinary solution devoid of electrolytic action: hence the changes of concentration in the electrolyte occur by diffusion in the ordinary manner with a diffusivity D given by

$$D = \frac{k_2 v_1 + k_1 v_2}{v_1 + v_2}.$$

As this relation holds good however slight the electric current may be, it may be presumed to hold in the limit where there is no current at all: therefore D is the coefficient of ordinary diffusion of the solution, and we have thus one physical relation involving k_1 and k_2 with the diffusivity and the electric data of ionic mobility.

On substitution for dn/dt from this equation of diffusion in either of the original differential equations, we obtain

$$\frac{k_2 - k_1}{v_2 + v_1} \frac{d^2 n}{dx^2} = \frac{d}{dx}\left(n \frac{dV}{dx}\right),$$

whence on integration

$$\frac{k_2 - k_1}{v_2 + v_1} \frac{dn}{dx} - n \frac{dV}{dx} = f(t).$$

Now we have

$$\frac{I}{e} = -(v_2 + v_1)\, n \frac{dV}{dx} + (k_2 - k_1) \frac{dn}{dx},$$

so that this integral merely reiterates the fact already implied in the equations, that the current I is uniform all along the solution. To obtain the expression of the law of conduction, we integrate the value of dV/dx given by it: thus

$$V' - V'' = \frac{k_1 - k_2}{v_1 + v_2} \log \frac{n''}{n'} + \frac{I}{(v_1 + v_2)\, e} \int_{x'}^{x''} \frac{dx}{n},$$

where $V' - V''$ represents the difference of potential, or electromotive force, between the two sections, at x' and x'', of the solution, and n'/n'' is the ratio of the concentrations at those points. If the first term on the right were absent, this equation would represent Ohm's law, the factor by which I is multiplied being the expression for the resistance between these sections in terms of the ionic mobilities. As things are, to retain Ohm's mode of statement the first term must be transferred to the left and combined with the electromotive force: in other words, we

see that change of concentration, along the direction of the current, from density ρ' to density ρ'', originates a backward or opposing electromotive force equal to

$$\frac{k_1 - k_2}{v_1 + v_2} \log_e \frac{\rho''}{\rho'},$$

where the subscript 1 refers to the cation. We have here another physical relation involving k_1 and k_2 with the observed electromotive forces of concentration.

These two relations suffice to independently determine k_1 and k_2 in terms of quantities directly measurable. As however on any view of electrolysis the relation $k_1/v_1 = k_2/v_2, = \beta$, must hold, there is thus a necessary relation between the diffusion coefficient and the electromotive force of concentration. But it was Nernst's great discovery that the actual values of these coefficients of migration k_1 and k_2, for both ions, are the same as follow from the hypothesis of independent mobilities of those ions, namely that β is equal to RT. The existence of this relation thus forms a logical demonstration that the anion and cation in dilute solutions diffuse under the influence either of variation of density or of electric force, approximately as if each were quite free of the other.

5. This argument corroborates, in a more direct manner, the one on which the principle of the independent mobilities of the anion and cation is usually founded, namely that in comparing dilute electrolytic solutions with non-electrolytic solutions, the osmotic pressure (or what comes to the same thing, the lowering of the vapour pressure or the depression of the freezing point) is twice as great for the former kind in comparison with the number of dissolved molecules. In arguing directly from this fact towards the independent mobility of anion and cation we require a *theory* to explain how it is that the osmotic pressure is connected only with the number of independent foreign nuclei in the solution, whether they are molecules or sub-molecules. The explanations that are usually given on this head are analogical and devoid of dynamical cogency. A valid theory however exists: but it is based on the theoretically rather recondite thermodynamic principle of

available energy, the relations of which to molecular theory form a delicate subject*. Thus the inference from the abnormality in the osmotic pressure to the independent mobilities of anion and cation is there logically so refined as to entail cautious handling: whereas all the considerations with which we have here been dealing refer directly to the diffusion of ions and their mobility under electric force, without any recondite molecular dynamics.

6. It is a striking circumstance that, on the hypothesis of independent mobility of the ions, the part of Faraday's law which asserts that electrochemical equivalents are liberated at the two electrodes, does not appear as a result of the mechanism of the electrolytic conduction, but is rather a constraint forced upon it from the outside, which forms the source of all the complication. Its cause must thus be sought in the nature of the conduction in the metallic part of the circuit: which points towards the view that in metals there is no diffusion of ions, but that they are passed on in a regular Grotthus'-chain fashion. This indication is strikingly at variance with the earlier ideas of the nature of metallic conduction.

See Ld Corr.

As regards the mode of electrolytic conduction, these results can be expressed in words independent of theoretical conceptions as follows. The electric force produces a drift of the anion and cation in opposite directions, with equal speeds in accordance with Faraday's law: at the same time it produces a uniform drift of the electrolyte across the solution towards the cathode, of which the velocity across any section corresponds to the passage of $\frac{v_1 - v_2}{v_1 + v_2}\frac{I}{2e}$ molecules of the electrolyte per unit area per unit time: this uniform drift of the dissolved substance is continually producing accumulation up against the cathode plate and abstraction from against the anode plate, which are simultaneously being relieved by spontaneous diffusion back again. As the ions diffuse at different speeds, whether electrolysis is going on or not, such changes of

* Cf. *loc. cit. ante*, *Phil. Trans.* 1897 A; or p. 287 *supra*.

concentration give rise to internal electromotive forces which prevent the electric gradient from being uniform across the solution: but the conduction always follows the law of Ohm. These principles have been demonstrated for the straight flow across an electrolyte in an ordinary cell: they clearly remain valid with slight modification of statement when the changes of concentration are not laminar, and the electric flow is in three dimensions. The velocity of convection of the electrolyte may be specified in a form independent of ions and their mobilities. In the first place, the available energy of the solution, estimated in von Helmholtz's manner by measuring its vapour tension, will give for the electromotive force arising from concentration an expression $A \log \rho''/\rho'$, where A is an experimental constant: then our present result is that an electric current I produces a velocity of drift of the electrolyte along with it amounting to $AI/2RTe$ molecules per unit time: and the rest involves only the laws of electric flow and of diffusion.

If v varies as $f(T)$ where T is the temperature, then k varies as $Tf(T)$, and the coefficient D of diffusion of the electrolyte through the solvent follows the latter law, while electromotive forces of concentration would be proportional to the temperature.

To remove the possibility that the phenomena thus described may be open to some other explanation that does not involve independent mobility of the ions, the visual method of experiment introduced by Lodge * was necessary: as applied by himself and by Whetham it appears to be decisive.

7. It is now a problem in the mathematics of diffusion (Fourier's linear conduction of heat) to start with a uniform solution, and a given electromotive force applied to it or a given current forced through it, and trace out the progress of the changes of concentration that are set up by the current, when the solution is supposed to be free from currents of mechanical convection.

* Brit. Assoc. Report, 1886: cf. Whetham's 'Solution and Electrolysis.'

The concentration n alters by simple diffusion according to the equation

$$\frac{dn}{dt} = D\frac{d^2n}{dx^2}, \text{ where } D = \frac{2\beta v_1 v_2}{v_1 + v_2},$$

so that D is the same whatever current be passing; and it only remains to specify the terminal conditions which hold at the electrodes. These may be obtained from the equations of § 2: thus at the cathode

$$\frac{dN_1}{dt} = \frac{I}{e}, \quad \frac{dN_2}{dt} = 0,$$

so that there

$$\frac{I}{e} = -nv_1\frac{dV}{dx} - k_1\frac{dn}{dx},$$

$$0 = -nv_2\frac{dV}{dx} + k_2\frac{dn}{dx};$$

leading to

$$\frac{dn}{dx} = -\frac{1}{2\beta v_1}\frac{I}{e} \text{ at the cathode,}$$

and similarly

$$\frac{dn}{dx} = \frac{1}{2\beta v_2}\frac{I}{e} \text{ at the anode.}$$

When the current I is maintained constant a particular integral is

$$n = c + bx - ax^2 - 2Dat,$$

where by the terminal conditions, the origin being taken at the anode and its distance from the cathode being l,

$$a = \frac{I}{2Del}, \quad b = \frac{I}{2\beta v_2 e}.$$

The general integral is obtained by adding an integral which makes I null everywhere and dn/dx null at both electrodes. When the effect of the initial conditions has died away, the ultimate state is that here given. Thus the concentration diminishes uniformly with the time all along the solution as the electrolysis proceeds: its gradient tends to a definite form irrespective of the value of the concentration itself, changing uniformly from $I/2e\beta v_2$ at the anode to $-I/2e\beta v_1$ at the cathode. The difference in the concentrations at the anode and cathode

is $\frac{Il}{2eD}\frac{v_1 - v_2}{v_1 + v_2}$; which is equal to Hittorf's difference produced initially per unit time divided by D/l*.

The gradient of electromotive force is determined by the equation for the current (p. 295), and is of complex character. The least number of molecules of the electrolyte which this steady state can contain is $Il^2/12\beta e$ multiplied by $2v_1^{-1} - v_2^{-1}$, or $2v_2^{-1} - v_1^{-1}$, according as v_1 is less or greater than v_2.

8. A different special case is that in which the circuit is broken, so that the current is null. Then

$$\frac{dN_1}{dt} = -\frac{dN_2}{dt}, \quad \frac{d^2N_1}{dxdt} = -\frac{dn}{dt},$$

leading directly to

$$\frac{dn}{dt} = \frac{k_1 v_2 + k_2 v_1}{v_2 + v_1}\frac{d^2n}{dx^2}$$

$$\frac{dV}{dx} = -\frac{k_1 - k_2}{v_1 + v_2}\frac{d}{dx}\log n.$$

The first equation gives for the coefficient of diffusion D the same formula as we have already found from the general analysis when a current is flowing: the second equation gives the same expression $\frac{k_1 - k_2}{v_1 + v_2}\log\frac{n''}{n'}$ for the electromotive force arising from concentration that has been already found from the general analysis.

This special case in fact formed the basis of Nernst's demonstration of his formulae. It may be noticed that here the state of concentration is not steady, the only possible steady state being one of uniform density: but the formula for the electromotive force is quite independent of what may be the law of concentration between the terminals when no current is flowing. It has been seen already that this remains true when a current is present.

9. An interesting application of these principles arises when an extraneous magnetic field H is established transverse

* When it is the applied electromotive force, instead of the current, that is kept constant, the quantities will vary exponentially with the time.

to the electric flow*. The individual ions will then be urged sideways with forces $e.v_1F.H$ and $e.v_2F.H$ both acting in the same direction, where F is the electric force driving the current. The aggregate of these forces will make up the Amperean transverse mechanical force acting on the electrolyte. But they have also an electromotive aspect: for they will tend to heap up the two ions sideways at different rates and thus produce electric separation leading by its statical action to a transverse electric force F'. With notation analogous to § 3, we have now, z denoting transverse measurement,

$$\frac{dN_1}{dt} = n_1v_1(F' + v_1FH) - k_1\frac{dn_1}{dz}$$

$$\frac{dN_2}{dt} = n_2v_2(-F' + v_2FH) - k_2\frac{dn_2}{dz},$$

where however dN_1/dt and dN_2/dt are here the drifts both measured along z positive. As before, n_1 and n_2 are so large that we can take them to be equal, say to n. Also $k_1 = \beta v_1$ and $k_2 = \beta v_2$, where $\beta = RT$. In the steady state dN_1/dt and dN_2/dt are both null: hence

$$\frac{\beta}{n}\frac{dn}{dz} = F' + v_1FH = -F' + v_2FH$$

so that $$F' = \tfrac{1}{2}(v_2 - v_1)FH$$

$$\frac{d}{dz}\log n = \frac{v_2 + v_1}{2\beta}FH.$$

This transverse electric force F' is uniform and independent of the concentration: thus it arises from a purely superficial electrification on the sides of the electrolyte. It is the force whose existence was suspected by Hall from considerations of the same nature as the above, though indefinite, and which was detected by him as a minute effect in metals. As the intensity of the current I is given by

$$I = en(v_2 + v_1)F$$

we have† $$F' = \frac{v_2 - v_1}{v_2 + v_1}\frac{IH}{2ne}.$$

* An investigation covering the more general case of partial ionisation is given by F. G. Donnan, *Phil. Mag.* Nov. 1898.

† Cf. *Phil. Trans.* 1894 A, p. 815.

Thus the coefficient of the Hall effect in a very dilute electrolytic solution should be $(v_2-v_1)/(v_2+v_1)\eta$, where η is the total electrochemical equivalent of the electrolyte per unit volume: it is $g/\beta\eta$, where the electromotive force due to change of concentration between densities ρ'' and ρ' is $g\log\rho''/\rho'$.

The change of concentration across the solution, given by the above value of $d\log n/dz$, might possibly be experimentally detected: it will not affect the resistance of the cell when the electric flow is in parallel lines; but if the lines of flow are not straight, for example if the electrodes are points instead of plates, the resistance between them will be minutely altered by a magnetic field. The alteration of the resistance of metallic bismuth by a transverse magnetic field does not appear to be of this nature, as it occurs in a thin wire.

10. This leads us on to consider whether an imposed magnetic field at right angles to the direction of the Earth's motion might not produce effects of electric separation in an electrolytic substance, whether carrying a current or not. Here the transverse electric force arising from the magnetism is vH, where v is the velocity of the electrolyte arising from the Earth's motion, the force being equal and opposite for the two ions: hence the equations are, when there is no current,

$$\frac{dN_1}{dt}=nv_1(F'+vH)-k_1\frac{dn}{dz}$$

$$\frac{dN_2}{dt}=nv_2(-F'-vH)-k_2\frac{dn}{dz}.$$

In the steady state dN_1/dt and dN_2/dt are null: so that

$$\frac{d}{dz}\log n=\frac{v_1}{k_1}(F'+vH)=\frac{v_2}{k_2}(-F'-vH).$$

As v_1/k_1 must be equal to v_2/k_2 on any theory of electrolysis, whether we adopt the hypothesis of independent ionic mobilities or not, it follows that $F'+vH$ is null. Thus as we might have anticipated, the total electric force inside an electrolytic substance partaking in the Earth's motion is strictly null whatever magnetic field be present, just as in a metallic

conductor: in other words there is a Hall effect F' which cancels the induced electrostatic field of force arising from the convection. When a current is present, the Hall effect will be diminished by this electrostatic field: but there will be no alteration in its galvanometric indications, because this field contributes no electromotive force round a circuit.

11. The problem for a solution of more than one electrolyte is much more complex. If we adopted the Williamson-Clausius hypothesis of mobile association of the ions, then in so far as each anion could adopt as a new partner only a cation of its own kind, the phenomena of the different electrolytes would be simply superposed; though even on that view there seems to be no reason why an anion may not recombine just as readily with one of the other cations,—such a reason if it existed must be presumably of the Grotthus'-chain type, but the links of the chain would become extremely weak when the solution is very dilute and therefore the molecules of the electrolyte very far apart. If we keep to the dissociation hypothesis, the ions will in a sufficiently dilute solution all be independently mobile, and it will at first sight no longer be necessary that the numbers of anions and of cations of the same electrolyte shall be the same in each element of volume: this will only be necessarily true of the aggregate, each ion counting proportionally to its valency. Here a complication enters, because in its electric aspect a p-valent ion is the same as p univalent ones superposed, while in its osmotic aspect it is only one: let us then simplify the conditions by treating it as p univalent ions, at the same time dividing the diffusion coefficient by p for each of them. Suppose there are members $n_1, n_2, n_3, \ldots$ per unit volume, of cations of various kinds, and $n_1', n_2', n_3', \ldots$ of anions of various kinds; we shall have equations of types

$$\frac{dN_1}{dt} = -n_1 v_1 \frac{dV}{dx} - k_1 \frac{dn_1}{dx}$$

and

$$\frac{dN_1'}{dt} = -n_1' v_1' \frac{dV}{dx} + k_1' \frac{dn_1'}{dx},$$

the force causing diffusion of each ion arising from its *partial* osmotic pressure; where

$$n_1 = -\frac{dN_1}{dx}, \quad n_1' = +\frac{dN_1'}{dx}, \quad \Sigma n = \Sigma n',$$

p being suppressed as above; while

$$\Sigma \frac{dN}{dt} + \Sigma \frac{dN'}{dt} = \frac{I}{e},$$

which is constant along the electrolyte.

Thus for the ions with subscript r, we have

$$\left(\frac{d}{dt} - k_r \frac{d^2}{dx^2}\right) N_r = v_r \frac{dN_r}{dx}\frac{dV}{dx},$$

$$\left(\frac{d}{dt} - k_r' \frac{d^2}{dx^2}\right) N_r' = -v_r' \frac{dN_r'}{dx}\frac{dV}{dx},$$

where $$k_r = \frac{\beta}{p_r} v_r, \quad k_r' = \frac{\beta}{p_r'} v_r'.$$

On substitution for N_r from this equation in the expression for I, there results a differential equation for V, of great complexity: the form of V being considered to be determined from that equation, the one just expressed will give the law of concentration of n_r. In a uniform solution of mixed electrolytes the current is found to be carried by them in the ratio of their conductivities, so that there is no preferential selection at the electrodes. Even for the case of only three ions the general results will be complex.

At each instant during the process of diffusion the current is given by

$$\frac{I}{e} + \frac{d}{dx}(\Sigma kn - \Sigma k'n') = -(\Sigma vn + \Sigma v'n')\frac{dV}{dx}:$$

this can be expressed in terms of Ohm's law by the statement that the variable concentration produces a back electric force of intensity*

$$\frac{d}{dx}(\Sigma kn - \Sigma k'n') \Big/ (\Sigma vn + \Sigma v'n'),$$

* On the whole subject see Planck, *Wied. Ann.* 39, 40, and 44.

the coefficient of $-dV/dx$, namely $e(\Sigma vn + \Sigma v'n')$, being the conductivity.

For a given law of fall of potential, *supposed independent of the time*, the value of N is of the type

$$N_r \propto F_r(x, v_r t), \quad N_r' \propto G_r(x, v_r' t),$$

where F and G are functional symbols, which will be the same for all values of r and r' respectively for which the valency is the same, because for all such k_r/v_r is the same. In the steady state that is ultimately attained by the system, dN_r/dt must be independent of x, so that $N_r \propto f(x) + A_r t$; thus as $n_r = -dN_r/dx$ the laws of concentration of all ions that are of the same valency will be the same in the steady state. If all the ions present have the same valency, the steady state of the system will therefore be obtained by superposing the steady states for the separate electrolytes.

12. The account given above of the changes of concentration produced by a current in different parts of an electrolyte supposes that the electrolyte is contained in a mass of solvent which is itself prevented from drifting on along with the dissolved substance: this condition holds good in all ordinary cases. But in electrolytic conduction through a narrow tube, opening into large masses of the solution at both ends, the drift of the molecules of the electrolyte through the tube will carry along the solvent as well, by the usual frictional agencies: and the current will thus be accompanied by an electric transpiration of the fluid through the tube. The question thus arises whether this agency is sufficiently powerful to produce effects comparable with the electric transpiration experimentally investigated by Quincke: it is however not reversible, being entirely frictional, so that an extraneous pressure driving the fluid through the tube cannot thus produce an electric current. Such phenomena of diffusion and transpiration will be much more definite in narrow tubes or pores than in wide spaces, because they will not then be very sensibly masked by mixing up of the liquid in mass owing to gravitational and thermal disturbances.

The easiest thing to determine is the osmotic pressure-head set up between the two ends of the tube when the transpiration is prevented. The gradient of pressure must be due to the extraneous forces acting on the contents of the tube, that is to the electric force $-dV/dx$ acting on the ions. Now the numbers of positive and negative ions are relatively practically the same, but there must be a very slight difference otherwise the force would be uniform all along: there is in fact a minute bodily electrification of density ρ given by $4\pi\rho = -K\nabla^2 V$, and the extraneous mechanical force acting on the fluid is thus $-\rho\dfrac{dV}{dx}$ or $\dfrac{K}{4\pi}\dfrac{dV}{dx}\nabla^2 V$. We may take it that the electric force is practically constant across the area of each section of the tube, so that this extraneous force is $\dfrac{K}{8\pi}\dfrac{d}{dx}\left(\dfrac{dV}{dx}\right)^2$, which amounts for the whole length l of the tube to a pressure-difference

$$\frac{K}{8\pi}\left\{\left(\frac{dV}{dx}\right)_2^2 - \left(\frac{dV}{dx}\right)_1^2\right\},$$

where the force $-dV/dx$ is in electrostatic units.

The mechanical forces thus indicated exist only in solutions variable as regards composition or cross-section, and are excessively minute compared with the observed electrolytic transpiration pressures*: such forces would be sensible in a highly charged condenser with leaking dielectric; the air currents produced by them in an air-condenser traversed by Röntgen radiation have been utilized by Zeleny† to trace the features of the ionization.

13. Certain thermo-electric phenomena in metallic circuits are also related to the present subject. Clausius was the first to theoretically connect the thermo-electric difference of potentials at the junction between different substances with the Peltier effect there situated. It was pointed out however by Lord Kelvin that the formula obtained by him, on the basis of Carnot's principle, was too simple for the facts, as it did not

* For von Helmholtz's theory, involving a layer of free ions near the wall of the tube as in frictional electrification, cf. Collected Papers, I, p. 876.

† *Proc. Camb. Phil. Soc.* 1898.

involve Cumming's phenomenon of thermo-electric reversal. This might arise from either of two causes, or from both: the whole thermodynamic procedure may be invalid because it is applied to a case in which degradation is continually going on, in the form of conduction of heat, along the same circuit which conducts the current, and of amount depending on the first power of the temperature-differences: or other thermo-electric effects may exist of which Clausius did not take account. It does not appear that the fundamental objection to the procedure can be safely ignored, considering that conductivity for heat is closely connected with conductivity for electricity*; but, waiving that, Lord Kelvin has assigned, as a cause of the discrepancy, what amounts to a convection of heat by the ions of the current, and such an action has been experimentally detected.

Suppose that in travelling from a place where the temperature is T to a place where it is $T + \delta T$, the positive ions of the current absorb heat equal to $s\delta T$ per unit electric charge, which is required to raise their mean kinetic energies by the amount corresponding to the rise of temperature δT, and that similarly the oppositely travelling negative ions give out $s'\delta T$: then the total absorption per unit quantity of electricity by a current travelling up the gradient of temperature is $\frac{1}{2}(s - s')\,\delta T$, or say $\sigma\delta T$, where σ has been named the 'specific heat of electricity' for the conductor and may be either positive or negative.

Let us then—ignoring the finite degradation by heat-conduction, but realizing that the electric flow may be made so slow that the electric degradation, proportional to the square of the current, is negligible, and that therefore the operations are certainly electrically reversible in Carnot's sense—apply the principle of energy and Carnot's principle to a circuit, formed of two metals and including as part of itself the dielectric of a condenser having these metals for its coatings, the temperature T varying from point to point

* It will be removed if the heat-conduction proceeds in *entire* independence of the electric current, except as regards the transfer of the ions the influence of which is reversible and is separately taken into account by the Kelvin coefficient. The electric cycle can moreover be completed in so short a time that the thermal transfer by ordinary conduction may possibly be neglected.

along the circuit. When the plates of the condenser are moved closer together without alteration of temperature its charge increases, as the difference of potential E between the plates remains constant; so that there is an electric flow round the circuit, and there is at the same time a gain of mechanical work and of available electric energy each equal to $\frac{1}{2}E\delta Q$, or in all E per unit total flow. Thus the plates of the condenser being at the same temperature T_2, we have, by the energy principle and Carnot's principle, considering unit electric flow round the circuit,

$$E = \Pi_1 + \int_{T_1}^{T_2} (\sigma - \sigma')\, dT$$

$$0 = \frac{\Pi_1}{T_1} + \int_{T_1}^{T_2} \left(\frac{\sigma}{T} - \frac{\sigma'}{T}\right) dT,$$

where Π_1 is the Peltier effect at the temperature T_1 of the junction of the metals, and σ, σ' are the 'specific heats of electricity' in them. Thus

$$\sigma - \sigma' = T_1 \frac{d}{dT_1}\left(\frac{\Pi_1}{T_1}\right),$$

$$E = \Pi - \int^{T_1} T \frac{d}{dT}\left(\frac{\Pi}{T}\right) dT = \int^{T_1} \frac{\Pi}{T}\, dT.$$

Hence for a temperature T of the junction, everything can be expressed in terms of the curve connecting the electromotive force E of the circuit with T, by the well-known simple relations

$$\frac{\Pi}{T} = \frac{dE}{dT}, \quad \frac{\sigma - \sigma'}{T} = \frac{d^2E}{dT^2}.$$

The Peltier effect appears, in the expression for E, in the form of an electromotive force at the junction*. The chemical mutual attractions of the molecules of the two metals across the inter-

* This follows on taking the temperature to be uniform. If however we adopted von Helmholtz's idea that each substance has a specific affinity for 'electricity' which varies with the temperature, and that the energies and entropies of the conductors in the system therefore involve terms proportional to their electric charges, but no other electric terms, we should arrive (cf. Parker, 'Thermodynamics' 1894 p. 260) at the result $\Pi = TdU/dT$, where U is the potential-difference at the junction, and there would be no gradient of potential along an unequally heated homogeneous wire. Doubtless there is intrinsic mutual available energy of the bodies and their charges, to be thus taken into account as a source of potential-difference; but it will depend on both the conductor and the surrounding medium because the charge is situated at their

face produce in fact a polar electric orientation of these molecules which gives rise to an abrupt potential-difference of contact equal to Π, and each electron e passing across the junction thus introduces an energy-effect $e\Pi$ which involves absorption or evolution of heat at that place in the Peltier manner. What then is the source of the other term in E, namely $\int(\sigma - \sigma') \, dT$? Thermodynamically it is involved in a convection of heat by the ions passing from a warmer to a colder part of the wire; and the mode in which it can thus arise may be put in evidence. For heat essentially consists largely in energy of molecular or atomic translational velocity: hence differences of effective ionic mobilities must in some degree enter here, and will have to be counteracted as in the electrolytic case by a slight bodily electric charge of free ions which will cause the back electromotive force necessary to keep the current uniform across all sections: this electromotive force is in Lord Kelvin's nomenclature $\int\sigma dT$. The mere temperature gradient could not, it may be held, produce a gradient of true contact potential-difference, for the mutual actions of molecules of the same kind cannot by orientating each other originate a residual polarity, inasmuch as any polarities there may be excited in a pair of them by their interaction will be equal and opposite: difference of molecular constitution is required to produce true contact potential-difference.

The same principles of ionic mobility point directly to the initiation of an electric force by the interaction of a magnetic field and a temperature gradient in a conductor, the direction of this force being at right angles to both these vectors and its magnitude depending on their vector product, as in the Hall effect; for the transfer of heat requires that, in the main, each molecule moves with rather greater speed down the gradient of temperature than up it. Such a force, and the converse phenomenon, have been actually detected by von Ettingshausen and Nernst*.

interface: it will in fact constitute a superficial distribution of energy, being a function of the state of the surface of the conductor: it thus indicates an additional and independent electromotive force, located at the surface of each conductor instead of at their junction, and constituting the main part of the voltaic potential-difference.

* See Riecke, 'Exp. Physik' II p. 327.

APPENDIX D

ON THE HISTORICAL DEVELOPMENT OF ATOMIC AND RADIANT THEORY

Fermat *on Least Time or Action*

"Synthesis ad Refractiones

"PROPOSUIT doctissimus Cartesius refractionum rationem experientiae, ut aiunt, consentaneam: sed, eam ut demonstraret, postulavit et necesse omnino fuit ipsi concedi, luminis motum facilius et expeditius fieri per media densa quam per rara, quod lumini ipsi naturali adversari videtur.

"Nos itaque, dum a contrario axiomate—motum nempe luminis facilius per media rara quam per densa procedere—veram refractionum rationem deducere tentamus, in ipsam tamen Cartesii propositionem incidimus. An autem contraria omnino via eidem veritati occurri possit ἀπαραλογίστως, videant et inquirant subtiliores et severiores Geometrae; nos enim, missa mataeotechnia, satius existimamus veritate ipsa indubitanter potiri, quam superfluis et frustrariis contentionibus et jurgiis diutius inhaerere.

"Demonstratio nostra unico nititur postulato: *naturam operari per modos faciliores et expeditiores.* Ita enim *αἴτημα* concipiendum censemus, non, ut plerique, *naturam per lineas brevissimas semper operari.*

"Ut enim Galilaeus, dum motum naturalem gravium speculatur, rationem ipsius non tam spatio quam tempore metitur, pari ratione non brevissima spatia aut lineas, sed quae expeditius, commodius, et breviori tempore percurri possint, consideramus."

Fermat, letter to M. de la Chambre, 1662:
in 'Œuvres' i, 1891, p. 173.

The Aether-theory of Huygens

To Huygens is due the credit of not merely originating an undulatory theory of light, but of expounding correct ideas of the general nature of the elasticity of a medium such as is required for the propagation of regular undulations*. He supposes that it is the very rapid agitation of the particles of luminous bodies, 'which swim in the aether,' that communicate the undulations to that medium. "L'agitation au reste des particules qui engendrent la lumiere doit estre bien plus prompte, et plus rapide que n'est celle des corps qui causent le son, puisque nous ne voyons pas que le fremissement d'un corps qui sonne est capable de faire naitre de la lumiere, de mesme que le mouvement de la main dans l'air n'est pas capable de produire du Son."

Then follows an explanation of the different modes of propagation of sound and light, which involves a remarkable conception of the kinetic origin of aereal pressure, much more vivid than anything given by Daniel Bernoulli†, as well as a correct view of the nature of the elasticity of homogeneous media, and the consequent uniformity of velocity of all pulses whether intense or weak. "Quant aux differentes manieres dont j'ay dit que se communiquent successivement les mouvemens du Son, et de la lumiere, on peut assez comprendre comment cecy se passe en ce qui est du Son, quand on considere que l'air est de telle nature qu'il peut estre comprimé, et reduit à un espace beaucoup moindre qu'il n'occupe d'ordinaire; et qu'à mesure qu'il est comprimé il fait effort à se remettre au large: car cela joint à sa penetrabilité, qui luy demeure non obstant sa compression, femble prouver qu'il est fait de petits corps qui nagent et qui sont agitez fort viste dans la matiere etherée, composée de parties bien plus petites. De sorte que la cause de l'extension des ondes du Son, c'est l'effort que font ces petits corps, qui s'entrechoquent, à se remettre au large, lorsqu'ils sont

* 'Traité de la Lumière,' written 1678, published 1690, Chapter I; Newton had calculated the velocity of sound and of waves on shallow water in the 'Principia,' 1686.

† 'Hydrodynamica,' sectio X, 1738, where Boyle's law was shown to follow from the kinetic hypothesis.

un peu plus serrez dans le circuit de ces ondes qu'ailleurs. Mais l'extreme vitesse de la lumiere, et d'autres proprietez qu'elle a, ne sçauroient admettre une telle propagation de mouvement, et je vais monstrer icy de quelle maniere je conçois qu'elle doit estre. Il faut expliquer pour cela la proprieté que gardent les corps durs à transmettre le mouvement les uns aux autres." Then he points out how a simple pulse is propagated along a row of glass or steel balls, in contact, by mutual collisions: that the essence of this action lies in the elasticity of the material which opposes resistance to any deformation and ultimately annuls it by resilience. That such deformation is the cause of the rebound of an elastic ball is seen by smearing it with grease: after rebound a circle of grease has been removed from it, and this circle is the larger the greater the velocity. Then follows a speculation as to the kinetic origin of the elasticity of the aether, which virtually makes the atom the core of a vortex-ring. "Mais quand nous ignorerions la vraye cause du ressort, nous voyons tousjours qu'il y a beaucoup de corps qui ont cette proprieté; et ainsi il n'y a rien d'étrange de la supposer aussi dans des petits corps invisibles comme ceux de l'Ether. Que si l'on veut chercher quelqu'autre maniere dont le mouvement de la lumiere se communique successivement, on n'en trouvera point qui conviene mieux que le ressort avec la progression égale, qui semble estre necessaire, parce que si ce mouvement se ralentissoit à mesure qu'il se partage entre plus de matiere, en s'éloignant de la source de la lumiere, elle ne pourroit pas conserver cette grande vitesse dans de grandes distances. Mais en supposant le ressort dans la matiere etherée, ses particules auront la proprieté de se restituer également viste, soit qu'elles soient fortement ou foiblement poussées; et ainsi le progrez de la lumiere continuera tousjours avec une vistesse égale."

It is explained that, as it is the rapid motions of the particles of water that render it more permeable and less resistant than sand, so an extremely brisk agitation of the particles may be the cause why the aether does not retard the planets. The circumstance that the aether can be expelled from a Torricellian vacuum by rise of the mercury is claimed as

a proof of its being able to pass with perfect facility among the molecules of matter: the fact that its undulations can set these molecules into vibration suggests their being constituted of smaller particles which would individually be more amenable to the disturbance.

The clear *aperçu* of the principle of wave propagation named after him, and his kinematic explanation of the laws of ordinary and double refraction, are well known : the extracts given above show that Huygens also possessed a remarkable intuition of the physical basis of the modern analysis of the phenomena of elasticity of solids and other media treated as continuous, as well as of the modern kinetic molecular theory of gases and liquids.

It is interesting to compare this prevision by Huygens of the nature of modern kinetic theories of matter, and indeed the whole tenour of his physical ideas as contained in the 'Traité de la Lumière,' with Cotes' polemic against the hypothesis of an all-pervading medium which is the main theme of the preface contributed by him to the second edition of the 'Principia' (1713). Huygens' cosmical views had however led him (notwithstanding the above extracts) to deny that gravitation could be an essential property of matter, though he agreed that large masses do in fact gravitate to each other in the Newtonian manner: and it was perhaps against his 'Discours de la Cause de la Pesanteur' (1690), which ascribed gravity to the pressure of the surrounding vortically moving aether, rather than the vaguer metaphysical ideas of Leibnitz, that Cotes' observations were mainly directed. The entirely reasonable attitude of Newton himself on this subject is illustrated by the extract next following, and by the well-known letters to Bentley. The somewhat reckless way in which the advocacy of a position can be pushed on both sides beyond rational limits, and the obvious difficulty in appreciating any merit in a point of view unfamiliar and at variance with the accustomed one, exhibited in this and similar instances, is a sufficient explanation of Newton's extreme reluctance to take part in such controversies.

Huygens was a thorough-going Cartesian, not in the sense which the term came to bear, as a believer in the special system

of vortices which Descartes tried to elaborate, but as an adherent of the dogma that substance cannot act where it is not, that all action of one body A on another body B at a distance from it must be capable of being definitely traced all the way across from A to B. It was this doctrine that stimulated him to the formulation and development of the wave theory of light. On the publication of the 'Principia' the same mode of thought led him to see the cause of gravitation between two bodies at a distance in some aethereal connexion extending across the intervening space, as to which he attempted an explanation of his own: he agreed that the simple law of inverse squares was established by the facts for the case of the heavenly bodies or other masses far apart: but he could not persuade himself that there was any likelihood that the aethereal connexion would be equivalent to a law of that degree of simplicity in the case of bodies very near together or of different parts of the same body. In fact the law of gravitation, as applying to the action "of every particle of matter on every other particle of matter," was a hypothesis* whose proof was to come ultimately from the results involved in it: the quantitative evidence then forthcoming being only the corroboration afforded by the fair accord between terrestrial gravity at the Earth's surface and at the distance of the Moon. Perhaps it is not too much to say that to this day the evidence that the law of gravitation is the *exact* law of inverse squares for moderate distances is of indirect character, except in so far as it is indicated by the fair accordance of the measurements of the constant of gravitation that have been made under various conditions. The most convincing argument is still founded on the consideration that the weight of a body does not depend on its orientation or position, thus showing that the transmission of gravitation cannot be modified by intervening matter: this can hardly be explained except on the hypothesis that the matter is of a discrete character and that its nuclei occupy very little space in the medium that is concerned in the gravitational propagation. This explanation again demands and is confirmed by the fact that the gravitational forces exerted by neighbouring

* For Newton's own statements cf. 'Principia,' lib. III, propp. 6, 7.

atoms do not sensibly interfere with each other,—as for example the disturbances in fluid arising from two pulsating spheres would do when their distance is of the order of their radii,—but are simply additive. Adjacent atoms do however exert mutual aethereal actions on each other, depending on the strains and motions involved in their structures; and their configurations must be themselves slightly disturbed thereby. The exactness of the law of conservation of weight implies that the mutual proximity does not modify these structures in any way that concerns gravitation; it follows that the separate sub-atoms, virtually point-nuclei, which constitute the material atom, gravitate independently without being affected by their orbital motions or the presence of neighbouring sub-atoms, all which is moreover exactly in keeping with the ascertained extreme rapidity of propagation of gravitational influence. When the phenomenon is thus resolved into attractions transmitted independently between atoms so small that any sensible distance is extremely great compared with their own dimensions, the validity of the extension of the simple astronomical law to all sensible distances becomes directly involved.

The notion that a mass is thus constituted of independent atoms can hardly in the light of the extracts given above have been foreign to Huygens' point of view : otherwise his difficulties would have been still more formidable. On the other hand, in the Queries at the end of Newton's 'Opticks' the attraction of gravitation is assigned to the pressure of an ambient medium*; so that considerations relating to its properties do not seem to have formed part of the reasons for Newton's belief† in an atomic constitution of matter.

The Aether-theory of Newton

Although Sir Isaac Newton was unable to understand that light propagated by waves could cast shadows, and for that reason felt compelled to fall back on projection rather than undulation in order to account for optical transmission, he yet

* 'Opticks' ed. 2, 1717, Query 21, p. 325.

† Cf. especially *loc. cit.* Query 31, pp. 350—382: for date, cf. Brewster's 'Life' II, p. 368.

made full use of the conception of an aether, active in chemical, thermal, and electrical phenomena, which by its undulations affected his moving 'corpuscles' so as to adapt them for reflexion and transmission at equidistant intervals. It was reserved for Young and Fresnel to explain this property of rectilinear propagation, as depending on the shortness of the waves, and thus definitely get rid of the extraneous machinery of corpuscles which Newton felt unable to avoid. An account of Newton's recorded pronouncements on the optical necessity of an aether is contained in Young's memoir 'On the Theory of Light and Colours'*: his conviction as to the necessity of a medium for the transmission of gravitation is emphatically expressed in the well-known Letters to Bentley.

Newton *to* Leibnitz, *Oct.* 1693, *on* Huygens' '*Discours sur la Cause de la Pesanteur*'

"Quae vir summus Hugenius in mea notavit ingeniosa sunt. Parallaxis solis minor videtur quam ipse statueram, et motus sonorum forte magis rectilineus est; at coelos materia aliqua subtili nimis implere videtur. Nam cum motus coelestes sint magis regulares quam si a vorticibus orirentur, et leges alias observent, adeo ut vortices non ad regendos sed ad perturbandos Planetarum et Cometarum motus conducant, cumque omnia coelorum et maris phaenomena ex gravitate solis secundum leges a me descriptas agente accurate quantum sentio sequantur, et natura simplicissima sit, ipse causas alias omnes abdicandas judicavi et coelos materia omni quantum fieri licet privandos, ne motus Planetarum et Cometarum impediantur aut reddantur irregulares. At interea si quis gravitatem una cum omnibus ejus legibus per actionem materiae alicujus subtilis explicuerit, et motus Planetarum et Cometarum ab hac materia non perturbatos iri ostenderit, ego minime adversabor."

Edleston's 'Correspondence of Sir Isaac Newton and Prof. Cotes...' p. 278.

Newton *on the Necessity of Atomic Theory*

"...Deinde ex his viribus per propositiones etiam mathematicas, deducuntur motus planetarum, cometarum, lunæ & maris. Utinam caetera naturæ phenomena ex principiis

* *Phil. Trans.* 1801: 'Lectures on Natural Philosophy,' quarto ed. Vol. II.

mechanicis eodem argumentandi genere derivare liceret. Nam multa me movent, ut nonnihil suspicer ea omnia ex viribus quibusdam pendere posse, quibus corporum particulæ per causas nondum cognitas vel in se mutuo impelluntur & secundum figuras regulares cohærent, vel ab invicem fugantur & recedunt: quibus viribus ignotis, philosophi hactenus naturam frustra tentarunt. Spero autem quod vel huic philosophandi modo, vel veriori alicui, principia hic posita lucem aliquam præbebunt."

Preface to 'Principia,' 1686.

Experimental Philosophy deals only with facts: yet an Aether is necessary for all physical actions

"...Rationem vero harum gravitatis proprietatum ex phænomenis nondum potui deducere, & hypotheses non fingo. Quicquid enim ex phænomenis non deducitur, *hypothesis* vocanda est; & hypotheses seu metaphysicæ, seu physicæ, seu qualitatum occultarum, seu mechanicæ, in *philosophia experimentali* locum non habent. In hac philosophia propositiones deducuntur ex phænomenis, & redduntur generales per inductionem. Sic impenetrabilitas, mobilitas & impetus corporum & leges motuum & gravitatis innotuerunt. Et satis est quod gravitas revera existat, & agat secundum leges a nobis expositas, & ad corporum cælestium & maris nostri motus omnes sufficiat.

"Adjicere jam liceret nonnulla de spiritu quodam subtilissimo corpora crassa pervadente, & in iisdem latente; cujus vi & actionibus particulæ corporum ad minimas distantias se mutuo attrahunt, & contiguæ factæ cohærent; & corpora electrica agunt ad distantias majores, tam repellendo quam attrahendo corpuscula vicina; & lux emittitur, reflectitur, refringitur, inflectitur, & corpora calefacit; & sensatio omnis excitatur, & membra animalium ad voluntatem moventur, vibrationibus scilicet hujus spiritus per solida nervorum capillamenta ab externis sensuum organis ad cerebrum & a cerebro in musculis propagatis. Sed hæc paucis exponi non possunt; neque adest sufficiens copia experimentorum, quibus leges actionum hujus spiritus accurate determinari & monstrari debent."

'Principia' Ed. 3, 1726; end of final scholium*.

* Cf. the detailed views in Query 31 at the end of 'Opticks' ed. 2, 1717.

Thomas Young *on an Electric and Optical Aether*

"That a medium resembling, in many properties, that which has been denominated ether, does actually exist, is undeniably proved by the phenomena of electricity; and the arguments against the existence of such an ether, throughout the universe, have been pretty sufficiently answered by Euler. The rapid transmission of the electrical shock shows that the electric medium is possessed of an elasticity as great as is necessary to be supposed for the propagation of light. Whether the electric ether is to be considered as the same with the luminous ether, if such a fluid exists, may perhaps at some future time be discovered by experiment: hitherto I have not been able to observe that the refractive power of a fluid undergoes any change by electricity...."

'Outlines of experiments and inquiries respecting Sound and Light,' *Phil. Trans.* 1800.

Sir H. Davy *on the Identity of Chemical Affinity and Electric Attraction*

"The relation of electrical energy to chemical affinity is however sufficiently evident. May it not be identical with it, and an *essential property of matter*?"

First Bakerian Lecture, 1806, section viii.

"I drew the conclusion [in 1806] that the *combinations and decompositions by electricity were referable to the laws of electrical attractions and repulsions*; and advanced the hypothesis '*that chemical and electrical attractions were produced by the same cause, acting in one case on particles, in the other on masses*'; *and that the same property, under different modifications, was the cause of all the phenomena exhibited by different voltaic combinations.*"

Lecture of date 1810, quoted in 'Life' 1836, by John Davy, Vol. I, p. 330.

C. F. Gauss *on the Law of Electrodynamic Action*

"I would doubtless have long ago published my researches, mainly of date 1834—1836, had there not, up to the time when I discontinued them, been wanting what I considered as the very keystone,

Nil actum reputans si quid superesset agendum,

namely the *deduction* of the law of force (which applies to the mutual actions of particles of electricity in relative motion as well as at rest) *from action not instantaneous* but propagated in time in a similar manner to light. This had not been reached by me: but so far as I remember I left the research at that time not without hope that it would probably be attained later, yet—if I remember right—with the subjective conviction that it would previously be requisite to form a working representation [construirbare Vorstellung] of the manner in which the propagation takes place."

Letter to W. Weber, Mar. 1845; translated from Gauss, 'Werke,' v, p. 629*.

Lord Kelvin *on the Nature of Atoms*

"......I can now tell the amount of the force [of attraction] and calculate how great a proportion of the chemical affinity is used up electrolytically before two such discs [of zinc and copper] come within $\frac{1}{1000}$ of an inch of one another, or any less distance down to a limit within which molecular heterogeneousness becomes sensible. This of course gives a definite limit for the size of atoms, or rather as I do not believe in atoms, for the dimensions of molecular structures."

Proc. Manchester Lit. and Phil. Soc., 1862.

Th. Graham *on the Constitution of Matter*

"To the preceding statements respecting atomic and molecular mobility, it remains to be added that the hypothesis admits of another expression. As in the theory of light we have the alternative hypotheses of emission and undulation, so in molecular mobility the motion may be assumed to reside

* Cf. Maxwell, 'Treatise' §§ 851, 861, 866.

either in separate atoms and molecules, or in a fluid medium caused to undulate. A special rate of vibration or pulsation originally imparted to a portion of the fluid medium [Roger Bacon's ὑλή] enlivens that portion of matter with an individual existence and constitutes it a distinct substance or element."

'Speculative Ideas respecting the Constitution of Matter,' *Phil. Mag.* 1864; Chemical and Physical Researches, p. 301,—cf. also the Introduction, by R. Angus Smith.

Fresnel *to* Arago, *on the Influence of the Earth's Motion on Optical Phenomena*

"Par vos belles expériences sur la lumière des étoiles, vous avez démontré que le mouvement du globe terrestre n'a aucune influence sensible sur la réfraction des rayons qui émanent de ces astres....

"Vous m'avez engagé à examiner si le résultat de ces observations pourrait se concilier plus aisément avec le système qui fait consister la lumière dans les vibrations d'un fluide universel. Il est d'autant plus nécessaire d'en donner l'explication dans cette théorie, qu'elle doit s'appliquer également aux objets terrestres; car la vitesse avec laquelle se propagent les ondes est indépendante du mouvement du corps dont elles émanent.

"Si l'on admettait que notre globe imprime son mouvement à l'éther dont il est enveloppé, on concevrait aisément pourquoi le même prisme réfracte toujours la lumière de la même manière, quelle que soit le côté d'où elle arrive. Mais il paraît impossible d'expliquer l'aberration des étoiles dans cette hypothèse: je n'ai pu jusqu'à présent du moins concevoir nettement ce phénomène qu'en supposant que l'éther passe librement au travers du globe, et que la vitesse communiquée à ce fluide subtil n'est qu'une petite partie de celle de la terre; n'en excède pas le centième, par exemple.

"Quelque extraordinaire que paraisse cette hypothèse au premier abord, elle n'est point en contradiction, ce me semble, avec l'idée que les plus grands physiciens se sont faite de l'extrême porosité des corps. On peut demander, à la vérité,

comment un corps opaque très-mince interceptant la lumière, il arrive qu'il s'établisse un courant d'éther au travers de notre globe. Sans prétendre répondre complètement à l'objection, je ferai remarquer cependant que ces deux sortes de mouvemens sont d'une nature trop différente pour qu'on puisse appliquer à l'un ce qu'on observe relativement à l'autre. Le mouvement lumineux n'est point un courant, mais une vibration de l'éther. L'on conçoit que les petites ondes élémentaires dans lesquelles la lumière se divise en traversant les corps peuvent, dans certains cas, se trouver en discordance lorsqu'elles se réunissent, en raison de la différence des chemins parcourus ou des retards inégaux qu'elles ont éprouvés dans leur marche ; ce qui empêche la propagation des vibrations, ou les dénature de façon à leur ôter la propriété d'éclairer, ainsi que cela a lieu d'une manière bien frappante dans les corps noirs; tandis que les mêmes circonstances n'empêcheraient pas l'établissement d'un courant d'éther. L'on augmente la transparence de l'hydrophane en la mouillant, et il est évident que l'interposition de l'eau entre les particules, qui favorise la propagation des vibrations lumineuses, doit au contraire être un petit obstacle de plus à l'établissement d'un courant d'éther; ce qui démontre bien la grande différence qui existe entre ces deux espèces de mouvemens.

"L'opacité de la terre n'est donc pas une raison suffisante pour nier l'existence d'un courant d'éther entre ses molécules, et l'on peut la supposer assez poreuse pour qu'elle ne communique à ce fluide qu'une très-petite partie de son mouvement.

"A l'aide de cette hypothèse, le phénomène de l'aberration est aussi facile à concevoir dans la théorie des ondulations que dans celle de l'émission; car il résulte du déplacement de la lunette pendant que la lumière la parcourt: or, d'après cette hypothèse, les ondes lumineuses ne participant point sensiblement au mouvement de la lunette, que je suppose dirigée sur le lieu vrai de l'étoile, l'image de cet astre se trouve en arrière du fil placé au foyer de l'oculaire d'une quantité égale à celle que parcourt la terre pendant que la lumière parcourt la lunette.

"Il s'agit d'expliquer maintenant, dans la même hypothèse,

comment la réfraction apparente ne varie pas avec la direction des rayons lumineux par rapport au mouvement terrestre."

Letter to Arago, *Annales de Chimie*, 1818.

The explanation given is, briefly, that refraction depends solely on difference of density of the aether, so that the density of the aether is proportional to μ^2: when a transparent body filled with this denser aether advances across the stagnant aether of free space with velocity v, a stream of aether must enter it in front and leave it behind, so that by the equation of continuity the aether inside it will advance but with velocity reduced by $\mu^{-2}v$*: and the light transmitted by this aether will partake of this velocity of advance $(1-\mu^{-2})\,v$. It is then verified that the laws of reflexion and refraction will, on this supposition, remain unaffected.

* Fresnel's explanation is obscure: he speaks of the enclosed aether being in part at rest and in part carried on along with the matter, and says that it may easily be seen that the velocity of light is increased by that of the centre of gravity of both parts. The above, which (p. 15) is the interpretation adopted by Stokes and by Maxwell, is doubtless his real meaning.

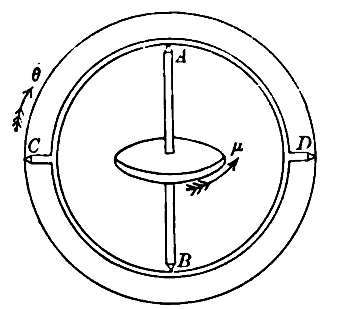
APPENDIX E

ON KINEMATIC AND MECHANICAL MODES OF REPRESENTATION OF THE ACTIVITY OF THE AETHER

Mechanical Models and Illustrations

"ALTHOUGH the Gaussian aspect of the subject, which would simply assert that the primary atoms of matter exert actions on each other which are transmitted in time across space in accordance with Maxwell's equations, is a formally sufficient basis on which to construct physical theory, yet the question whether we can form a valid conception of a medium which is the seat of this transmission is of fundamental philosophical interest, quite independently of the fact that in default of the analogy at any rate of such a medium this theory would be too difficult for development. With a view to further assisting a judgment on this question, it is here proposed to describe a process by which a dynamical model of this medium can be theoretically built up out of ordinary matter,—not indeed a permanent model, but one which can be made to continue to represent the aether for any assignable finite time, though it must ultimately decay. The aether is a perfect fluid endowed with rotational elasticity; so in the first place we have—and this is the most difficult part of our undertaking—to construct a material model of a perfect fluid, which is a type of medium nowhere existing in the material world. Its characteristics are continuity of motion and absence of viscosity: on the other hand in an ordinary fluid, continuity of motion is secured by diffusion of momentum by the moving molecules,

which is itself viscosity, so that it is only in motions such as vibrations and slight undulations where the other finite effects of viscosity are negligible, that we can treat an ordinary fluid as a perfect one. If we imagine an aggregation of frictionless solid spheres, each studded over symmetrically with a small number of frictionless spikes (say four) of length considerably less than the radius*, so that there are a very large number of spheres in the differential element of volume, we shall have a possible though very crude means of representation of an ideal perfect fluid. There is next to be imparted to each of these spheres the elastic property of resisting absolute rotation; and in this we follow the lines of Lord Kelvin's gyrostatic vibratory aether. Consider a gyrostat consisting of a flywheel spinning with angular



momentum μ, with its axis AB pivoted as a diameter on a ring whose perpendicular diameter CD is itself pivoted on the sphere, which may for example be a hollow shell with the flywheel pivoted in its interior; and examine the effect of imparting a small rotational displacement to the sphere. The direction of the axis of the gyrostat will be displaced only by that component of the rotation which is in the plane of the ring; an angular velocity $d\theta/dt$ in this plane will produce a torque measured by the rate of change of the angular momentum, and therefore by the parallelogram law equal to $\mu d\theta/dt$ turning the ring round the perpendicular axis CD, thus involving a rotation of the ring round that axis with angular acceleration $\mu/i\,.\,d\theta/dt$, that is with velocity $\mu/i\,.\,\theta$, where i is the aggregate moment of inertia of the ring and the flywheel about a diameter of the wheel. Thus when the sphere has turned through a small angle θ, the axis of the

* The use of these studs is to maintain continuity of motion of the medium without the aid of viscosity; and also (§ 4) to compel each sphere to participate in the rotation of the element of volume of the medium, so that the latter shall be controlled by the gyrostatic torques of the spheres.

gyrostat will be turning out of the plane of θ with an angular velocity $\mu/i\,.\,\theta$, which will persist uniform so long as the displacement of the sphere is maintained. This angular velocity again involves, by the law of vector composition, a decrease of gyrostatic angular momentum round the axis of the ring at the rate $\mu^2/i\,.\,\theta$; accordingly the displacement θ imparted to the sphere originates a gyrostatic opposing torque, equal to $\mu^2/i\,.\,\theta$ so long as $\mu/i\,.\int\theta dt$ remains small, and therefore of purely elastic type. If then there are mounted on the sphere three such rings in mutually perpendicular planes, having equal free angular momenta associated with them, the sphere will resist absolute rotation in all directions with isotropic elasticity. But this result holds only so long as the total displacement of the axes of the flywheels is small: it suffices however to confer rotatory elasticity, as far as is required for the purpose of the transmission of vibrations of small displacement through a medium constituted of a flexible framework with such gyrostatic spheres attached to its links, which is Lord Kelvin's gyrostatic model* of the luminiferous working of the aether. For the present purpose we require this quality of perfect rotational elasticity to be permanently maintained, whether the disturbance is vibratory or continuous. Now observe that if the above associated free angular momentum μ is taken to be very great, it will require a proportionately long time for a given torque to produce an assigned small angular displacement, and this time we can thus suppose prolonged as much as we please: observe further that the motion of our rotational aether in the previous papers is irrotational except where electric force exists which produces rotation proportional to its intensity, and that we have been compelled to assume a high coefficient of inertia of the medium, and therefore an extremely high elasticity in order to conserve the ascertained velocity of radiation, so that the very strongest electric forces correspond to only very slight rotational displacements of the medium: and it follows that the arrangement here described, though it cannot serve as a model of a field of steady electric force lasting for ever, can

* Lord Kelvin, *Comptes Rendus*, Sept. 1889: 'Math. and Phys. Papers,' III, p. 466.

yet theoretically represent such a field lasting without sensible decay for any length of time that may be assigned.

"It remains to attempt a model (cf. Part I, § 116) of the constitution of an electron, that is of one of the point-singularities in the uniform aether which are taken to be the basis of matter, and at any rate are the basis of its electrical phenomena. Consider the medium composed of studded gyrostatic spheres as above: although the motions of the aether, as distinct from the matter which flits across it, are so excessively slow on account of its great inertia that viscosity might possibly in any case be neglected, yet it will not do to omit the studs and thus make the model like a model of a gas, for we require rotation of an individual sphere to be associated with rotation of the whole element of volume of the medium in which it occurs. Let then in the rotationally elastic medium a narrow tubular channel be formed, say for simplicity a straight channel AB of uniform section: suppose the walls of this channel to be grasped, and rotated round the axis of the tube, the rotation at each point being proportional for the straight tube to $AP^{-2} + PB^{-2}$*: this rotation will be distributed through the medium, and as the result there will be lines of rotational displacement all starting from A and terminating at B: and so long as the walls of the channel are held in this position by extraneous force, A will be a positive electron in the medium, and B will be the complementary negative one. They will both disappear together when the walls of the channel are released. But now suppose that before this release the channel is filled up (except small vacuous nuclei at A and B which will assume the spherical form) with studded gyrostatic spheres so as to be continuous with the surrounding medium; the effort of release in this surrounding medium will rotate these spheres slightly until they attain the state of equilibrium in which the rotational elasticity of the new part of the medium formed by their aggregate provides a balancing torque, and the conditions all round A or B will finally be symmetrical. We shall thus have created two per-

[* This is corrected *infra*.]

p.328

manent conjugate electrons A and B; each of them can be moved about through the medium, but they will both persist until they are destroyed by an extraneous process the reverse of that by which they are formed. Such constraints as may be necessary to prevent division of their vacuous nuclei are outside our present scope; and mutual destruction of two complementary electrons by direct impact is an occurrence of infinitely small probability. The model of an electron thus formed will persist for any finite assignable time if the distribution of gyrostatic momentum in the medium is sufficiently intense: but the constitution of our model of the medium itself of course prevents, in this respect also, absolute permanence. It is not by any means here suggested that this circumstance forms any basis for speculation as to whether matter is permanent, or will gradually fade away. The position that we are concerned in supporting is that the cosmical theory which is used in the present memoirs as a descriptive basis for ultimate physical discussions is a consistent and thinkable scheme; one of the most convincing ways of testing the possibility of the existence of any hypothetical type of mechanism being the scrutiny of a specification for the actual construction of a model of it.

"An idea of the nature and possibility of a self-locked intrinsic strain, such as that here described, may be facilitated by reference to the cognate example of a *material* wire welded into a ring after twist has been put into it. We can also have a closer parallel, as well as a contrast; if breach of continuity is produced across an element of interface in the midst of an incompressible medium endowed with *ordinary material rigidity*, for example by the creation of a lens-shaped cavity, and the material on one side of the breach is twisted round in its plane, and continuity is then restored by cementing the two sides together, a model of an electric doublet or polar molecule will be produced, the twist in the medium representing the electric displacement and being at a distance expressible as due to two conjugate poles in the ordinary manner. Such a doublet is permanent, as above; it can be displaced into a different position, at any distance, as a strain-form, without the medium

moving along with it; such displacement is accompanied by an additional strain* at each point in the medium, namely, that due to the doublet in its new position together with a negative doublet in the old one. A series of such doublets arranged transversely round a linear circuit will represent the integrated effect of an electric polarization-current in that circuit; they will imply irrotational linear displacement of the medium round the circuit after the manner of vortex motion, but this will now involve elastic stress on account of the rigidity. Thus with an ordinary elastic solid medium, the phenomena of dielectrics, including wave-propagation, may be kinematically illustrated; but we can thereby obtain no representation of a single isolated electric charge or of a current of conduction, and the laws of optical reflexion would be different from the actual ones. This material illustration will clearly extend to the dynamical laws of induction and electromagnetic attraction between alternating currents, but only in so far as they are derived from the kinetic energy; the law of static attraction between doublets of this kind would be different from the actual electric law."

Phil. Trans. 1897 A, pp. 209—212.

1. This description of an ideal (supernatural) construction for electrons in a rotational aether requires correction as regards one point. The line integral of the rotation that has to be imparted to the walls of the canal AB is equal at each cross-section to the surface integral of the normal component of the rotational displacement of the aether over a surface abutting on it and enclosing either electron: it is therefore constant all along the canal, whether the latter is straight or curved, instead of proportional to $AP^{-2} + PB^{-2}$ as above stated. Thus if the canal is of uniform circular section, the rotational displacement of its walls is uniform all along it.

This circumstance allows a development of the analogy, which will further illustrate the origin of the mechanical attraction between two electrons. It is a well-known device in mechanical construction, to use a flexible wire of great torsional rigidity to transmit rotation from one shaft to another

[* See footnote, p. 336.]

not in line with it, by clamping the ends of the wire to the ends of the shafts so that it forms an elastic connexion between them. Now instead of filling up our ideal canal in the aether by a filament of aether, let us suppose it filled up by such a wire, of infinite torsional rigidity, and in continuous connexion with the surrounding aether. Each time any cross section C of this wire is rotated round its axis by an impressed torque, the rotation is transmitted all along the wire, and thence to the aether alongside it; and two complementary electrons are thus developed at its ends A and B. On releasing this section C the rotation undoes itself, and these terminal electrons disappear. This arrangement constitutes an elastic system devoid of any intrinsic stress such as was previously implanted in the system by filling up the canal with aether itself; for it becomes free from stress on releasing the wire. We should therefore be in a position to point directly to the proximate cause of the attraction of one electron on the other. It is to be found in the tangential tractions which the surrounding aether exerts on the surface of the wire, which form a system of forces statically equivalent, by virtue of the principle of virtual work, to an attraction between its ends.

We can in this way imagine the aether with its contained electrons as mathematically dissected into an elastic medium devoid of intrinsic strain, by connecting each positive electron with a complementary negative one by means of such an elastic material wire AB in continuous connexion with the aether, to which has been imparted at any cross section C the amount of rotation proper to maintain the intensities of the electrons. When the wire has disappeared and the electrons at A and B are permanently constituted by filling up its place with aether, the possibility of thus specifying a proximate cause of the mechanical attraction between the electrons has also in a sense disappeared. But just as the exploration of the relations of a cyclic analytical function requires the introduction of cross-cuts or barriers in its domain to render account of the cyclic character, so the complete elucidation of the dynamics of a medium involving cyclic intrinsic strain requires the introduction of ideal canals or tubes connecting the strain-centres, through

operation on which this strain may be considered as implanted in the medium*. We can even consider the tractions exerted on the surface of such a tube of strain as statically transmitted to the electrons at its ends just as if it included the wire of the illustration. Even when the wire is present the amount of the attraction is most easily determined by application of the principle of Energy: this method remains available when it is absent, so long as it is definitely recognized that the Energy principle, or more generally the Action principle, is a fundamental dynamical method whose application is not limited to the class of cases in which we are able to *describe* the activity of the medium in terms of familiar processes of direct elastic transmission. Although the simultaneous representation of the two kinds of existing forcive, aethereal stress and material attractions, thus transcends the usual elementary notion of elastic propagation, they yet appear alongside each other in the development of the dynamical formulation of the medium in terms of the principle of Action, which is prior to any model whatever, and is moreover logically required, unless we are content to view the medium as a system of relations in space and time represented by differential equations devoid of dynamical significance. We thus conclude, along with von Helmholtz, that there is no resting place in general dynamical theory or explanation, short of the Action foundation. The content of this principle, as applied to continuous media, is in

* When the medium is thus completely specified, the line integration in Stokes' theorem of curl will contain integrals round the sections of these tubes where they cross the sheet. But it is only the ends of the tubes that are determinate; hence to obtain a definite result we must (as in i, p. 90, in which the fluxion dots should be deleted) apply the theorem only to the change, indicated there by the Δ, that results from small displacements of the existing electrons; each displacement of an electron is formally equivalent to the establishment of a tube of strain connecting its old with its new position, and whenever this tube crosses the sheet a correcting term is required in the formula.

A convenient mode of developing the electrodynamics of material media would be to replace the translational displacement of each electron by the local rotational displacement of the aether itself which is its constitutive equivalent as regards that medium; the problem can then be treated by methods of continuous analysis applied to free aether. Cf. *Camb. Phil. Trans.*, Stokes' Jubilee volume, 1900.

various ways wider than the conception of simple elastic transmission, which is the case that is most familiar in the more easily analyzed classes of physical phenomena. We might for example have an energy function involving second as well as first differential coefficients of the displacements†, in which case disturbances would still be transmitted by the medium, but not by the agency of simple elastic stress definable in terms of surface tractions alone: it is only the extreme shortness of the range of molecular action compared with the size of the element of mass that is just sensible to our powers of observation, that debars this case from being a practical one.

In point of history, the dynamics of elastic propagation was first developed in a somewhat inexact way by Navier and Poisson, and attempts were subsequently made to establish it on an incomplete molecular foundation by Cauchy and others. But there was no reliable foothold obtainable even for this simple case until Green, by one of his strokes of genius, summarily included the whole matter under the Action principle. Reference to a transmitting medium was previously instructive by way of general illustration, for example in physical optics, but before the use of this principle by Green and by MacCullagh there was no sufficiently exact and general formulation of its possible modes of activity. It is in this way that the Action principle is prior even to the exact development of a theory of simple elastic transmission: and it is thus not surprising that it forms the most suitable basis when the transmitting medium is constituted in a more complex manner.

2. The subjects discussed in this book have in the main been treated without any hypothesis as to the structure of the nucleus of an electron. In a preliminary stage of the development of this theory, the analogy of an electron to a conductor carrying an electric charge suggested that the nucleus of an electron might be treated as a minute spherical region in which the aether is effectively devoid of elasticity: but this is not an essential or even probable feature. The illustration above given,

† Cf. pp. 207, 356 footnote.

of a nucleus of intrinsic strain in an elastic solid, indicates that what is essential is the concentration of 'beknottedness' in the small volume of the medium which constitutes the nucleus, which would thus correspond to a small volume-electrification. Such an intrinsic strain-form*) is mobile through the medium, without thereby originating any new distribution of stress around it, because it is only rotation and not deformation of the aether that calls out elastic reaction; and this free mobility is an essential element in the theory. But the analysis into independent strains and rotations, on which it rests, requires that both strain and rotation shall be very small; thus the inertia of the medium must be very great, and each nucleus must be so constituted that the intrinsic rotations involved in its structure are so small that they can everywhere be treated as differential rotations, which is demanded by the linearity of the scheme of equations as well as by the mobility of the nucleus.

The dynamical scheme developed in Chapter VI is however based solely on the application of the method of Action to a medium uniform throughout all space, specified by the Lagrangian function $T - W$ of p. 84, and involving in its constitution mobile poles or electrons which by their aggregation form a representation of matter, at any rate in those respects in which it interacts with the aether. In that scheme the effective aethereal displacement represented by (ξ, η, ζ) need not be defined: it is not necessary (and it was not there intended) to assume it to be a translational displacement. The scheme thus stands on a formally definite basis independently of any knowledge of the type of disturbance that (ξ, η, ζ) represents: and it has not as yet been shown to be too narrow to represent the field of general physical actions.

In the model or illustration of the working of the aether which has been here described, this disturbance (ξ, η, ζ) is taken to represent translational displacement of the element of aether originally situated at the point (x, y, z). The medium is then one whose elasticity is purely and solely rotational. One object of the gyrostatic mode of representation above explained is to render the idea of rotational elasticity more familiar and more

*) The nucleus being of unknown constitution, its mobility has to be assumed; that of its strain form is already secured.

easily grasped, by illustrating it from the properties of an actual medium which could theoretically be constructed from ordinary matter. It is also of use towards allaying scruples that naturally arise as to the legitimacy of assuming a set of abstract properties of a type not met with in matter under ordinary conditions, and therefore liable to the suspicion of being somehow self-contradictory or in opposition to formally necessary dynamical principles: but though an actual model of such a medium forms a valuable and forcible illustration, the argument is logically complete without it. Such a gyrostatic model has no claim to be more than an illustration of the properties of the aether, for an aether of the present type can hardly on any scheme be other than a medium, or mental construction if that term is preferred, prior to matter and therefore not expressible in terms of matter.

This more special hypothesis that takes the variable (ξ, η, ζ) in p. 84 to be proportional to actual translational displacement, involves on the other hand a question of direct fact, as to which there are physical means of inquiry: its further consideration is therefore called for. It has been explained that, whatever be the character of the vector (ξ, η, ζ), the facts as regards the influence of the Earth's motion on optical phenomena, as well as the linear character (p. 96) of the electrodynamic equations, require that the aether shall be practically stagnant. On the present hypothesis this vector, whose time-gradient represents magnetic force, must therefore be equal to the translational displacement of the medium multiplied by a very large numerical constant. There is in fact no phenomenon known which is inconsistent with the ultimate simplification of passing analytically towards a limit, by taking the translational displacement to be indefinitely small and this multiplier indefinitely great.

The question suggests itself, as to what inducement there is to specify (f, g, h) as of the type of rotational displacement at all, seeing that the theory developes itself without any reference to the type of disturbance which this vector represents. The only motive is that the number of unconnected hypotheses, which dynamically cannot be independent, is thereby reduced: the possibility of the intrinsic elastic structure of an electron,

and that of its free mobility, will be in the more indeterminate theory two new assumptions, both of unaccustomed character: while on the more special view they are both merged as corollaries in the single interpretation of the relations of the aethereal medium, so that the scheme proceeds on that basis alone. But in the case of a mind to which this simplification does not appeal, either as an elimination of a group of hypotheses that cannot from the nature of the case be independent and are so liable to the possibility of being inconsistent with each other, or else as an assistance to vivid apprehension of the relations*, the argument can proceed without any necessity for its adoption.

3. It is not merely convenient, but is also necessary, for the mathematical analysis of a medium involving electrons, to transform the independent variable as in Chapter VI from (ξ, η, ζ) to (f, g, h), if we are to evade an analytical dissection of the medium by means of the strain-tubes of § 1. For the former variable can represent only the change of state of the medium arising from the displacement of the electrons: the primordial creation of these electrons required also displacements of this type (ξ, η, ζ), which involved discontinuous processes, but left no trace after the discontinuities of the

* It is desirable to further emphasize that these representations are illustrative, not essential: it may be held that they are too imperfect to be useful, without giving up anything essential in the theoretical formulation of the phenomena. In ultimate logic any physical representation is in fact a mental construction or analogy, designed to relieve the mind from the intangible and elusive character of a complex of abstract relations. It thus involves a correlation of a range of phenomena with something else that can be constructed either actually or mentally. It is however unreasonable to suppose that two things not the same can have *complete* identity of relations: on the other hand the universal employment of such ideal pictures constitutes evidence that they are legitimate and powerful aids to knowledge. Our mental image, whether abstract or illuminated by a model, cannot ever be completely identical with the complex of phenomena which it represents, though it is capable of continued approximation thereto. The essential problem is to determine in each case how deep the correspondence extends: if it is found to extend into unforeseen properties and lead to the recognition or prediction of new relations in the field of the actual phenomena, its propriety within due restrictions is usually considered to be vindicated: it is in fact in this way that most advances of knowledge arise. Cf. pp. 68–71; also Hertz's 'Mechanik,' Introduction.

medium had been healed except the presence of the rotational strain belonging to them. Thus it is only the latter variable (f, g, h), which is proportional to the strain, that can express the complete state of the medium, including the positions of the electrons as the intrinsic poles of the strain. When the Action is expressed in terms of this latter variable, its variation analyzes the forcive of the system into torques acting on the elements of volume of the aether and forces acting on the electrons, which are both supplemented by internal stresses, determined only as to type, arising from the Lagrangian multipliers of Chapter VI, these stresses being involved in maintaining continuity in the internal constitution of the system and being determined ultimately by the condition that they do so. The mode in which the torque thus acting on the medium is propagated by it appears subsequently in this procedure from the analysis of the resulting equations: but the forces acting on the electrons, though in a sense transmitted by the medium, are not propagated across it at all. Yet it is these latter forces alone,—of which the aggregates give rise to the electric force altering electric distributions and the mechanical forces acting on material bodies that are electrically excited or transmit electric currents,—that are the primary realities as regards our perceptions, the strains propagated in the aether itself being wholly inferential. The Energy-principle, or more generally, the Action-principle, as thus developed, is of wider scope than the idea of simple step-by-step propagation which represents the results of applying it to a homogeneous elastic medium devoid of singularities. This recognition of dynamical action between systems, arising otherwise than by direct propagation of elastic stress between them, however in no way implies that the aether is not the sole medium of transmission: it may for example be recalled that the mutual actions of vortex rings in perfect fluid are not propagated in time across the fluid, though they take place by its intervention*.

* It is the intrinsic strain-form alone that constitutes the electron; and it is a fundamental postulate that the form can move from one portion to another of the stagnant aether somewhat after the manner that a knot can slip along a cord.

4. The essential contrast between thoroughgoing constitutive theories of the aether, like the vortex-atom theory and the one above sketched, and the usual theories of radiation which ascribe to the aether an extremely minute density compared with matter, is that on the former view the aether is fundamental, and its properties must be adapted to be consistent, by themselves alone, with the whole range of physics, whereas on the latter view we have an independent dynamics of matter treated as fundamental, and the aether must be arranged so as but slightly to interfere with it. The latter view virtually identifies aether with a species of matter. Its difficulties become conspicuous as soon as we admit the modern theory that the energy of a magnetic field is distributed in the surrounding region of free space and is constituted of aethereal kinetic energy: if we assume very small inertia, this must involve either velocities of translation of the aether, of altogether impossible magnitudes, or else a cellular structure in which the energy exists in some way as energy of gyrostatic rotation, so that the magnetic force is some kind of kinematic vector which is not translatory. That being assumed, the

If this form is taken as an entity, so that its position is part of the specification of the system, then the dynamical analysis introduces forces acting on it: it is possible that the origin of these forces might be further analyzed by aid of a deeper knowledge of the constitution of the system, but at present it suffices to consider them as effectively an ultimate datum. In a rotational aether an electron thus mobile has been constructed: its displacement from A to B involves rotation of the medium around the successive elements of its path in such manner that there is no additional strain produced. Whereas when an intrinsic strain-form is implanted in an elastic solid, of the type indicated on p. 327 but with its nucleus extended over a minute volume so that the intrinsic deformation thus inserted by supernatural processes of rupture and healing is nowhere finitely discontinuous, it cannot in general (as there assumed) be removed to another position even by imposing additional strains in the medium, because the discontinuities involved in its creation cannot be entirely annulled by imposing any continuous strain: thus there cannot be phenomena of freely mobile strain-forms connected with a solid medium. In the case of the electron an additional hypothesis is involved that the nucleus does not break up: this transcends the rotational scheme, which stops short of the constitution of the nucleus. It is to be noticed that in the case of a strain-nucleus in a solid, the strain-field may be considered as created by tractions applied to a surface surrounding it; but in the case of an electron a strain-tube travelling out from this surface is also required.

difficulty is transferred to conceiving a mechanism by which the vibrations of simply material atoms can transfer energy into a medium so differently constituted: whereas in the former type of theory the whole of the energy of the vibrations of the atoms belongs to the aether because the atoms themselves belong to it. These discussions can however be to some extent deferred, if we are willing to admit without explanation the scheme of equations derived in Chapter VI from the form of energy-function for the aether, supposed stagnant, which is there postulated, in combination with the principle of Least Action and, as a corollary, with an atomic structure of matter, involving electrons in its specification.

The electron theory required by the electrodynamics of steady currents

"According to the type of theory which considers a current system to be built up of physical current-elements of the form $(u, v, w)\,\delta\tau$, the energy associated with an element of volume $\delta\tau$, as existing in the surrounding field and controlled by the element, is

$$T = (Fu + Gv + Hw)\,\delta\tau.$$

"The ponderomotive force acting on the element will be derived from a potential energy function $-T$, by varying the coordinates of the material framework: it must in fact consist, per unit volume, of a force

$$\left(u\frac{dF}{dx} + v\frac{dG}{dx} + w\frac{dH}{dx},\quad u\frac{dF}{dy} + v\frac{dG}{dy} + w\frac{dH}{dy},\quad u\frac{dF}{dz} + v\frac{dG}{dz} + w\frac{dH}{dz}\right),$$

and a couple

$$(vH - wG,\quad wF - uH,\quad uG - vF),$$

the former being derived from a translational, the latter from a rotational virtual displacement of the element. We may simplify these expressions by taking the axis of z parallel to

the current in the element $\delta\tau$, so that u and v become null; then we have

$$\text{a force } \left(w\frac{dH}{dx},\quad w\frac{dH}{dy},\quad w\frac{dH}{dz}\right)$$

and a couple $(-wG,\ wF,\ 0)$.

"According to the Ampère-Maxwell formula, there should be simply a force at right angles to the current, specified by the general formula

$$(vc - wb,\quad wa - uc,\quad ub - va),$$

which becomes for the present special axes of coordinates

$$\left\{-w\left(\frac{dF}{dz}-\frac{dH}{dx}\right),\quad w\left(\frac{dH}{dy}-\frac{dG}{dz}\right),\quad 0\right\}.$$

"The forcive at which we have here arrived thus differs from the Ampère-Maxwell one by

$$\text{a force } \left(w\frac{dF}{dz},\quad w\frac{dG}{dz},\quad w\frac{dH}{dz}\right)$$

and a couple $(-wG,\ wF,\ 0)$:

these are equivalent to forces acting on the ends of each linear current element, equal at each end numerically to $(wF,\ wG,\ wH)$ per unit of cross section, positive at the front end and negative at the rear end. They are thus of the nature of an internal stress in the medium, and are self-equilibrating for each circuital current and so do not disturb the resultant forcive on the conductor as a whole due to the field in which it is situated. From Maxwell's stress standpoint they would form an equilibrating addition to the stress-specification in the conductor which is the formal equivalent of the electrodynamic forcive.

"According to the Ampère-Maxwell formula, the forcive on an element of a linear conductor carrying a current is at right angles to it, so that the tension along the conductor is constant so far as that forcive is concerned. The traction in the direction of the current arising from the above additional stress, would introduce an additional tension, equal to the current multiplied

by the component of the vector potential in its direction, which is not usually constant along the circuit, and so may be made the subject of experimental test with liquid conductors, as it would introduce differences of fluid pressure. There will also be an additional transverse shearing stress which should reveal itself in experiments on solid conductors with sliding contacts.

"In particular these additional forces should reveal themselves in the space surrounding a closed magnetic circuit, where the ordinary Amperean force vanishes because the magnetic field is null; in that case (F, G, H) may be interpreted as the total impulsive electric force induced at any point by the making of the circuit. Professor G. F. Fitzgerald has devised an experiment in which the behaviour of a thread of mercury carrying a strong current and linked with a complete magnetic circuit was closely observed when the circuit was made and broken. No movement was detected, whereas, when the magnetic circuit was incomplete, the ordinary Amperean forces were very prominent. According to the above analysis, the two types of forcive should be of the same order of magnitude in such a case: the result of the experiment is therefore against this theory. A like negative result has also attended an experiment by Professor O. J. Lodge, in which he proposed to detect minute changes of level along the upper surface of a uniform mercury thread by an interference arrangement on the principle of Newton's rings: when the current was turned on, the section of the thread became more nearly circular owing to the mutual attractions of the different filaments of the current, but there was no alteration in the direction of its length."*

Phil. Trans. 1895 A, pp. 698—700.

* Then follows a verification that the same result is deducible from the expression of Neumann and von Helmholtz for the mutual electrokinetic energy of two current elements, namely

$$\iota_1 \delta s_1 \iota_2 \delta s_2 \left\{ \frac{\cos(\delta s_1 \cdot \delta s_2)}{r_{12}} + \frac{d^2 f(r_{12})}{ds_1 ds_2} \right\}.$$

As regards however the phenomena of ordinary electrodynamics, which involve velocities and alternations slow compared with those of radiation, the

energy may all be considered as attached to the electrons, and everything may be deduced from the expression

$$e_1 v_1 e_2 v_2 \left\{\frac{\cos(ds_1 \, . \, ds_2)}{r_{12}} + \frac{1}{2}\frac{d^2 r_{12}}{ds_1 \, ds_2}\right\}$$

for the mutual electrokinetic energy (cf. *Phil. Trans.* 1894 A, p. 812) of two electrons e_1 and e_2 moving with velocities v_1 and v_2 in the directions of ds_1 and ds_2. This would lead to a different value for the electric force (the force acting on an electron) from that given by Weber's formula $\frac{1}{2} e_1 e_2 r_{12}^{-1} (dr_{12}/dt)^2$; but it must give the same results (cf. Maxwell, 'Treatise' §§ 856—860) for the induced total electromotive force driving a current around a circuit and for the mechanical force on an element of a conductor carrying a current.

APPENDIX F

MAGNETIC INFLUENCES ON RADIATION AS A CLUE TO MOLECULAR CONSTITUTION

The Zeeman effect

1. THE most direct and definite experimental indication towards the intimate structure of a molecule, hitherto obtained, has been the effect of a magnetic field on the character of its free periods of vibration, discovered by Zeeman and largely anticipated as to its general character from theoretical considerations by Lorentz and others.

If we regard the molecule as constituted of a system of ions, of various effective masses denoted by m and electric charges denoted by e, revolving round each other under their mutual electric and other forces, then when a steady magnetic field H is impressed in the direction (l, m, n), their equations of motion are modified into the type

$$\begin{aligned} m\ddot{x} - m\kappa\,(n\dot{y} - m\dot{z}) &= X \\ m\ddot{y} - m\kappa\,(l\dot{z} - n\dot{x}) &= Y \\ m\ddot{z} - m\kappa\,(m\dot{x} - l\dot{y}) &= Z, \end{aligned}$$

where $\kappa = eH/m$ when e is in electromagnetic units; (X, Y, Z) being the force acting on the ion arising from the configuration of the molecule or other causes.

If the system of ions, free from extraneous magnetic force, is referred to axes steadily rotating round the direction (l, m, n) with angular velocity ω, the formula for the component acceleration is altered from $\ddot{x}$ to

$$\ddot{x} - 2\omega\,(n\dot{y} - m\dot{z}) - \omega^2 x + \omega^2 l\,(lx + my + nz).$$

When ω is taken equal to $\frac{1}{2}\kappa$, and therefore ω^2 is negligible, the resulting equations referred to the moving axes become identical in form with the equations in the magnetic field. Hence if κ is the same for all the ions the two states of motion are identical: in other words if e/m is the same for all the ions, the effect of the impressed magnetic field H is simply to impose a rotation with angular velocity $\frac{1}{2}e/m \,.\, H$, around its axis, on some undisturbed motion of the system. This however requires that the specification of (X, Y, Z) is the same with regard to the moving axes as with regard to the fixed ones,—in other words, that the field of force acting on the electrons is symmetrical with respect to the axis of the impressed magnetic field, in so far as it does not arise from mutual forces depending only on the configurations of the moving system*.

The condition thus introduced requires that e shall have the same sign for all the moving ions: but it will be approximately fulfilled if there are additional ions for which e/m is small in comparison with the constant value that obtains for the others. We may for example suppose the charges to be the same for all the ions, and the effective masses of the positive ones to be large compared with those of the negative ones which must be themselves equal: then under their mutual forces, the velocities of the positive ones will be the smaller, inversely in the order of the ratio of their masses, and the value of κ for them will also be the smaller in the same ratio. We may still refer the motion of the system, when the magnetic field is applied, to the rotating axes; but it will now be necessary to impress forces on the positive ions in order to keep them in position. These forces will be small compared with the forces exerted by the magnetic field on the negative ions, and their effect will also be smaller on account of the greater masses on which they act. Thus we may consider the influence of the magnetic field as equivalent to a uniform rotation of the system, if we superpose on that rotation a much smaller disturbance due to these forces acting on the more massive positive ions. If the positive

* Cf. *Phil. Mag.* Dec. 1897.

ions are not more massive in this manner, this simple representation of the effect of the magnetic field as a rotation around its axis will not hold good. In no case can the negative ions be treated as moving independently of each other, for the electric forces between them are 'among the strongest of the forces of chemical affinity.'

It is to be noticed that in any case this rotation is not simply superposed on that orbital configuration which existed before the magnetic field was established. As the moving ions are supposed to be negative, that would in fact imply that all the molecules are polarized paramagnetically by the field. Consider however the ideal limiting case in which the field is impressed instantaneously: the velocities of the ions will remain continuous through that instant: hence the undisturbed orbit on which the rotation is imposed will be that corresponding to the positions of the ions at the instant, but with initial velocities reduced by removal of the velocities arising from the rotation thus imposed. The change of orbit thus involved will also introduce polarization, in the main of a diamagnetic character. In actuality the establishment of the magnetic field is a very slow process compared with the orbital periods, so that the readjustment of the orbits and the establishment of their rotation will be comparatively very gradual.

It appears incidentally that the conception of paramagnetism, which considers it to be due to orientation of the molecule as a whole by the magnetic field, as if it were a rigid system, is not valid except as a very rough illustration. Indeed otherwise its magnetic polarization would reach a limit if time enough were available, so that the magnetic coefficient per unit mass of a gaseous medium would increase very sensibly with diminution of density and consequent increase of free path of its molecules*. Moreover it appears from piezoelectric phenomena that each molecule has a mean intrinsic electric moment, so that orientation of any regular kind would introduce electric as well as magnetic polarization, whereas a process of the nature here

* The interesting opinion hazarded by Maxwell, 'Treatise' ii § 844, as to what would constitute experimental demonstration of the existence of Amperean currents in the molecule, would on this view be considerably modified.

described would not do so. The great magnitude of dielectric coefficients compared with magnetic coefficients is explained by the large charges of the ions on which the electric force acts, compared with the small effective currents on which the magnetic force acts. The exceptionally great magnetic coefficients of iron, nickel, and cobalt at ordinary temperatures may possibly be explained as an effect of molecular cohesion or grouping: the magnetic field may alter the conditions of a molecular group, which then adjusts itself to the constrained conformation: then the field can act afresh, to be followed by another readjustment of the group: and this cumulative adjustment by a creeping action may proceed a long way. It is in fact the case that when a ring even of the softest iron has been magnetized longitudinally by a current, the magnetism is retained until it is shaken out of the iron by mechanical or other disturbance: moreover in the rapid oscillatory field of Hertzian waves the magnetization has not time to get established at all, while elastic molecular processes like dielectric polarization are fully operative*.

2. We have now to consider the effect of a magnetic field on the radiation emitted by the molecule.

In the first place, if each ion described a steady closed orbit, the radiation sent out from the molecule could be resolved into rectangular components each of them exactly periodic, and therefore consisting by Fourier's theorem of a fundamental spectral line and its harmonics. As the harmonics of spectral lines do not actually occur, it follows either that there are no closed steady orbits or that the steady motions of a molecular system do not originate sensible radiation: reason in favour of the latter alternative has been already given (§ 151), which would not be affected by a rotation imposed on the molecule.

* The great solvent and ionizing powers of water and some other liquids have been commonly connected with their abnormally high dielectric coefficients: as these constants become normal at very low temperatures, or with very rapid alternation of the field as in optics, it may not be fanciful to suggest that the common cause of both properties may be found in facility for loose molecular aggregation.

When the steady state is disturbed, the effect will be, by the general theory of small oscillations, to superpose a series of elliptic harmonic inequalities, of different periods, on the steady motions of each ion, each of which would give rise to the radiation constituting a spectral line. The effect of the magnetic field on each such elliptic vibration would be a rotation, superposed on the rotation which it would produce in the steady orbital system: this will destroy its simple harmonic character. The elliptic vibration may however be decomposed into a linear component parallel to the axis of rotation, and an elliptic one transverse to that axis: the latter is equivalent to a circular transverse vibration together with a linear one, while for this linear one may be substituted two equal circular ones in opposite directions: thus in all we have a linear component parallel to the axis, and two circular ones of different amplitudes around it, all of the same period. These three components are differently affected as to period by the rotation, but in such way that they all remain simple harmonic: thus the magnetic field resolves each spectral line into a triad, with the features of polarization, linear and circular, that are involved in the above statement.

The observed perfect circular polarization of the outer lines of the Zeeman triplets, when viewed along the magnetic field, proves that the corresponding permanent types of vibration in the molecules are exactly circular. If they were merely elliptic with a common direction of rotation they would not compensate each other in the various molecules so as to produce an average circular effect, because the intensities of the fortuitously distributed radiations from the various molecules are additive: they would thus produce circularly polarized light accompanied by ordinary light of the same order of intensity. If we now drop the restriction of κ to the same value for all the effective ions of the molecule, this feature will give a clue towards a more general representation of the facts. This is desirable because that restriction requires that the difference of frequencies of the Zeeman constituents should be the same for all those lines in a spectrum which come from the same vibrating system,—which is in general far from being the case if the molecule constitutes a single vibrating system, although there

is ground for the belief that the difference is constant for the lines forming a series, thus agreeing with the view that these lines have a common origin.

It remains to consider the character of the force (X, Y, Z). It will be sufficient to take the surrounding aether as at each instant in an equilibrium conformation, so that these forces are of the nature of statical stresses and motional forces transmitted between the various ions, and thus arise from an energy function depending on the configuration and motion of those ions alone: for a disturbance in the aether can travel over about 10^3 diameters of the molecule during the period of a single vibration. Moreover the motional forces between the ions are negligible compared with the disturbing force of the impressed magnetic field: for they are of the order of magnitude of $e_1 e_2 \ddot{x}_1 / r_{12}$ and $e_1 e_2 \dot{x}_1^2 / r_{12}^2$. Now to gain an idea of the order of $\dot{x}$, let us consider the simple case of two equal electrons $+e$ and $-e$ describing circular orbits round each other under their mutual attraction with velocity v: then $mv^2/\frac{1}{2}r = C^2e^2/r^2$, while Zeeman's measurements give $e/m = 10^7$ *: hence taking r to be 10^{-8} and e to be 10^{-21} we obtain $v = 10^{-3}C$: the orbital period thus comes out of the same order as the periods of ordinary light, which affords some presumption that the general trend of this mode of representation is valid. With these estimates the ratios of the motional forces of type $e^2\dot{x}^2/r^2$, the statical forces of type C^2e^2/r^2, and the disturbing forces of the impressed magnetic field of type $e\dot{x}H$, are of about the same order as the ratios of 10^{-6} to unity to $10^{-9}H$. Thus the forces of the impressed magnetic field are more important than the motional forces between the ions when H exceeds 10^3; while they are both so small that their effects can in all cases be taken as additive.

3. To obtain a general representation of the facts as free as possible from hypothesis, it is convenient to take the axis of z parallel to the impressed magnetic field, as this will permit of

* Being of the same order as the values deduced for the masses of cathode particles from determinations of their velocities and charges, and their behaviour in magnetic and electric fields, by J. J. Thomson, cf. *Phil. Mag.* Nov. 1899, and confirmed by later measures by Kaufmann, Simon, Wiechert, and others, *Wied. Ann.* 1899.

the introduction of coordinates such that each of them specifies a permanent type of vibration. For it is clear that circular harmonic vibrations around that axis are represented by

$$x \pm \iota y = Ae^{\iota(pt+\alpha)},$$

where p is positive, and α is chosen so that A is the real amplitude, the upper and lower signs corresponding to right-handed and left-handed sense respectively. Thus the suitable coordinates for each ion are (ξ, η, z) where

$$\xi = x + \iota y, \qquad \eta = x - \iota y,$$

so that

$$2x = \xi + \eta, \qquad 2y = -\iota(\xi - \eta),$$

$$2\frac{d}{d\xi} = \frac{d}{dx} - \iota\frac{d}{dy}, \quad 2\frac{d}{d\eta} = \frac{d}{dx} + \iota\frac{d}{dy}.$$

Now the dynamical equations of this ion are

$$\begin{aligned} m(\ddot{x} - \kappa\dot{y}) &= X \\ m(\ddot{y} + \kappa\dot{x}) &= Y \\ m\ddot{z} \quad\quad &= Z \end{aligned}$$

where, by the reasons above stated, (X, Y, Z) is derived from a static potential function W so that

$$(X, Y, Z) = -k\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)W:$$

if the forces are wholly of electric origin k is the electric charge of the ion, but this need not at present be assumed. On transformation these equations become

$$\begin{aligned} \ddot{\xi} + \iota\kappa\dot{\xi} &= -\frac{2k}{m}\frac{dW}{d\eta} \\ \ddot{\eta} - \iota\kappa\dot{\eta} &= -\frac{2k}{m}\frac{dW}{d\xi} \\ \ddot{z} \quad\quad &= -\frac{k}{m}\frac{dW}{dz}, \end{aligned}$$

in which W is expressed in terms of ξ, η, z. There are as many of these sets of equations as there are ions, and k/m as well as κ may be different in the various sets.

The system will be a 'cycloidal' one, i.e. its vibration will be compounded of simple harmonic principal types, provided W

is a quadratic function: and the theory of gyrostatic cycloidal systems* shows that the periods of these types are all real or pure imaginary, being all real when W is essentially positive. It follows that an impressed magnetic field does not introduce dissipative influences into the motions in the molecule.

If $2\pi/(p_1, p_2, \ldots)$ denote the free periods, the general state of vibration of the system is represented by sets of equations of type

$$\xi = A_1 e^{\iota p_1 t} + A_2 e^{\iota p_2 t} + \ldots$$
$$\eta = B_1 e^{\iota p_1 t} + B_2 e^{\iota p_2 t} + \ldots$$
$$z = C_1 e^{\iota p_1 t} + C_2 e^{\iota p_2 t} + \ldots$$

in which $A_1, A_2 \ldots$ are arbitrary complex constants, to each of which the others belonging to the same period are proportional, in complex ratios, so that each period represents a definite mode or type of vibration. From these equations the values of x, y, z can be expressed: and a real state of motion will be derived by separating out the real (or the imaginary) part in the result. The component of each principal vibration in the plane of xy will usually be of elliptic harmonic character on account of the relation between the constants. The condition necessary and sufficient to enable these principal vibrations to be circular is that the system of equations break up into three sets, one of which involves only the ξ coordinates, another only the η coordinates, and the third only the z coordinates. Then they are of type

$$\xi = A_1 e^{\iota p_1 t} + A_2 e^{\iota p_2 t} + \ldots$$
$$\eta = A_1' e^{\iota p_1' t} + A_2' e^{\iota p_2' t} + \ldots$$
$$z = C_1 e^{\iota q_1 t} + C_2 e^{\iota q_2 t} + \ldots$$

where the constants $A_1', A_2' \ldots$ are independent of $A_1, A_2, \ldots$. When the former constants are null the equations represent the inter-connected group of right-handed circular vibrations, of

* Cf. Thomson and Tait, *Nat. Phil.* Ed. 2, §§ 345 i—345 xxviii: Routh, 'Dynamics' Vol. ii, §§ 310—319, or 'Essay on Stability' 1877, p. 78.

periods $2\pi/(p_1, p_2, \ldots)$: when the other constants are null, they represent the inter-connected group of left-handed circular vibrations of periods $2\pi/(p_1', p_2', \ldots)$. These groups are entirely independent of each other: an impressed vibratory or other stress, of circular type, will excite only the group belonging to its own sense of rotation.

The condition that the equations thus form three independent systems is that W is quadratic of the form

$$\Sigma B_{rs}\xi_r\eta_s + f(z_1, z_2, \ldots)$$

in which the suffixes now refer to different ions, and r may be the same as s. On changing back from ξ, η to x, y it will appear that this form of W is the most general quadratic function that is invariant as regards rotation of the axes of coordinates around that of z. Now this axis of z, being that of the impressed magnetic field, may be any axis in the molecule: hence W must be invariant with respect to all systems of rectangular axes, which restricts it to the form

$$\begin{aligned} W &= -\tfrac{1}{2}\Sigma A_{rs}\{(x_r - x_s)^2 + (y_r - y_s)^2 + (z_r - z_s)^2\} \\ &\qquad\qquad + \Sigma B_{rs}(x_r x_s + y_r y_s + z_r z_s) \\ &= -\tfrac{1}{2}\Sigma A_{rs}\{(\xi_r - \xi_s)(\eta_r - \eta_s) + (z_r - z_s)^2\} \\ &\qquad\qquad + \tfrac{1}{2}\Sigma B_{rs}(\xi_r\eta_s + \xi_s\eta_r + 2z_r z_s). \end{aligned}$$

We have tacitly been taking (x, y, z) to be the actual linear coordinates of the ion; the potential energy W is then simply the most general quadratic function of these coordinates that depends on their mutual configuration alone, so that no restriction is really involved. But (x, y, z) may and probably will represent linear deviation from some state of steady motion*. If then the potential energy relative to the steady motion is of the above form, the principal types of vibration in a magnetic field will be circularly polarized in the two directions around the axis, and linearly polarized along it, as required, the right-handed types forming by themselves an independent system, and also the left-handed. This latter condition is necessary to account for the independent propagation of right-handed and left-handed circular wave-trains in rotational media.

* Cf. Routh, 'Dynamics' Vol. ii, § 111.

Even in the absence of any conception of the nature of the steady motion in the molecule, on which the vibration is superposed, we are perhaps entitled to restrict W to this form: for the total potential energy must be a function of configuration only; it, or rather the modified Lagrangian function, is divided into a part belonging to the steady motion, which does not involve the vibrating coordinates at all, and a quadratic function of the latter; and it is to be expected that this latter part will, by itself, remain unchanged in form when the axes of coordinates are altered.

The period equation of the right-handed group of vibrations will be, if ϵ denote m/k,

$$\begin{vmatrix} -\epsilon_1 p^2 - \epsilon_1\kappa_1 p + \Sigma A_{1r}, & C_{12}, & C_{13} \ldots \\ C_{21}, & -\epsilon_2 p^2 - \epsilon_2\kappa_2 p + \Sigma A_{2r}, & C_{23} \ldots \\ C_{31}, & C_{32}, & \ldots \ldots \end{vmatrix} = 0$$

where $C_{rs} = C_{sr}$: that of the left-handed will be derived by changing the sign of κ, that of the rectilinear group by making κ null. In the absence of a magnetic field, a system of parallel linear vibrations of the ions, in any direction, will constitute an independent group, and all such systems will be identical.

This scheme is more general than the previous one which supposed κ to be the same for all molecules, in that the separation of the Zeeman constituents of the various spectral lines can be all different and arbitrarily specified: it is less general in that it only applies to the vibration about an unknown steady motion, whereas the former applied to the total motion of the system.

If the period equation for p^2, in the absence of a magnetic field, have equal roots, the magnetic field will usually separate them into two pairs of circular Zeeman vibrations and one linear vibration: while all three vibrations would be doubled, if the magnetic field can sensibly modify the steady motion on which they are superposed, so as to alter slightly the coefficients in the quadratic expression for W: such modification would however also cause displacement of the middle line of a simple Zeeman triplet.

This form of the potential energy of the disturbance makes each molecule optically isotropic, the polarization being proportional and parallel to the electric force whatever be its direction*. Thus when a wave-train is passing across the medium, each molecule is polarized exactly in the direction of the electric force. If on the other hand the molecules had aeolotropic quality, their orientations being irregular and changing from time to time, it would appear that as regards the vibratory polarity thus fortuitously induced in directions perpendicular to the inducing field each molecule might act as an independent secondary source of radiation, so that the wave-train would thus be subject to rapid damping.

In an electric theory of optical dispersion, the constants connecting the induced polarization of the molecule with the molecular coordinates† would also in that case be averaged constants belonging to a large aggregate of molecules orientated in all directions with regard to the inducing force, it being assumed that an effectively differential element of volume in the wave-train can be large enough to contain a very large number of molecules: the isotropy of the medium would then arise from this process of averaging. On the other hand, if each molecule is optically isotropic, the double refraction arising from crystalline structure or mechanical strain would be due entirely to the arrangement of the molecules in space. The intrinsic permanent electric polarity in the molecule, which is revealed by piezoelectric phenomena, is not involved in optical propagation. It is perhaps questionable whether the relations of precise polarization in the light diffracted by extremely minute particles or molecular aggregates in the atmosphere would be maintained, if the individual molecules were sensibly aeolotropic in their dielectric relations.

* Thus Kerr finds, *Phil. Mag.* 1895, that in the double refraction of a liquid dielectric which is induced by an electric field, it is only the light polarized so that its electric vibration is along the field that has its velocity affected.

† *e.g.*, the constants $c_1, c_2, \ldots c_1', c_2', \ldots$ of *Phil. Trans.* 1897 A, p. 238.

Relation between the Faraday and Zeeman effects

4. Under the condition κ constant of § 1, it has been seen that the effect of an impressed magnetic field H on a molecule is to force the steady conformation which constitutes it to rotate with small uniform angular velocity ω equal to $eH/2m$. When a wave-train of circularly polarized light is traversing the medium along the direction of the field, the aethereal vibration consists, at each cross-section, of bodily rotation of the aether, with very minute amplitude that varies harmonically along the train, but with very great angular velocity Ω that is uniform all along it. Moreover it has been seen (p. 346) that, the wave-length being about 10^3 molecular diameters, the reaction of each molecule on the wave-train depends sensibly only on the configuration of its ions. It follows that a wave-train of angular velocity $\Omega \pm \omega$ maintains the same series of configurations with regard to the molecules when the magnetic field is impressed as one with Ω does when it is absent, the + and − signs corresponding to the cases in which Ω, ω are in the same or opposite directions. Thus the reaction of the molecules bears the same proportion to the aethereal stresses maintaining the wave-train in both cases: and the velocities of propagation are therefore affected to the same extent in both cases, so that they are equal. Thus the velocity of propagation corresponding to circularly polarized light of period $2\pi/\Omega$ in the magnetic field H will be $V \mp \frac{dV}{d\Omega}\omega$, where $\omega = eH/2m$, and V is its velocity in the absence of the field, the sign varying according as it is right-handed or left-handed. Now if V_1 and V_2 are the velocities of the right-handed and left-handed components of an incident plane-polarized train of period $2\pi/\Omega$, the rotation of the plane of polarization in a length l of the medium will be $\frac{1}{2}\Omega\left(\frac{l}{V_1} - \frac{l}{V_2}\right)$, the latter factor being the difference of times of transit: in the present case it is thus $\frac{l\Omega}{V^2}\frac{dV}{d\Omega}\omega$, so that the rotatory power of the medium is $\frac{\Omega}{V^2}\frac{dV}{d\Omega}\frac{e}{2m}$. If λ

is the wave-length of the light in a vacuum and n its index of refraction in the medium, $V = C/n$ and $\Omega = 2\pi C/\lambda$, thus the rotatory power is $\frac{e}{2mC}\lambda\frac{dn}{d\lambda}$. It has been shown by H. Becquerel*, by whom this formula was brought forward, that it is in general a good approximation to the observed order of magnitude, while it well represents the relation of the rotation to the wave-length for creosote and sulphide of carbon. It would however make the coefficient positive for all media in which the dispersion is in the normal direction. If we suppose the dispersion of a medium to be controlled by a single absorption band representing a single free molecular period, or by a number of such bands for all of which the Zeeman constant is the same, the formula should apply exactly: otherwise it cannot be more than a rough indication. There is one important case in which it is always practically exact: for light of period close to a free period of the molecules the dispersion is anomalous and is controlled by that free period alone: the rotatory power in that neighbourhood is therefore proportional to $\lambda dn/d\lambda$ and is thus abnormally great and rapidly varying, being in opposite directions on the two sides of the free period†.

The Faraday effect of dispersional type

5. It appears from this discussion that magneto-optic rotation is a kinetic phenomenon related to the free periods

* *Comptes Rendus*, Nov. 1897.

† *Proc. Camb. Phil. Soc.* Mar. 1899. The discoverers of this abnormal rotation, Macaluso and Corbino, have obtained the result (*Rend. Lincei*, Feb. 1899) that it varies as $\lambda^{-1}dn/d\lambda$, by assuming that the magnetic field introduces a proportionate change in the wave-length instead of in the frequency: for the case discussed by them this is practically equivalent to Becquerel's formula, though it would not represent the state of affairs all along the spectrum.

Cf. the converse procedure of Voigt (*Ann. der Phys.* 1900, p. 390) which introduces the rotational terms into equations of propagation already containing simple dispersional terms so that there is an absorption band, the result being that this band is tripled: when there is a frictional term in the dispersion the tripling is asymmetric.

of the molecules, and not at all to their mean polarization in a steady electric field: it is therefore of dispersional character. Thus any attempt, such as that made in § 129, to extend to magneto-optic rotation the considerations by which Clausius and Lorentz established a relation between the mean refractive index and the density of the substance, cannot be expected to succeed; for the rotational term in the electric polarization is connected with the free periods as well as with the force.

Cognate considerations apply as regards the similar inquiry (§ 133) relating to intrinsic optical rotation. It has there been already remarked (cf. footnote) that the only type of rotational term that is allowable is one that could not exist in a steady electric field: thus that effect also depends on the periods of the vibrations, and so must be dispersional, although in that case no immediate physical representation of the origin of the term has suggested itself.

Thus we should expect optical rotatory power, intrinsic as well as magnetic, to be a function of both the dispersion and the density of the medium.

Direct determination of Optical Rotation

6. The general relations of the rotation of the plane of an optical vibration may be immediately inferred from the form obtained in Chapter XII for the relation connecting the electric polarization and the force. The general equations of the electrodynamic field are of form

$$\nabla^2 P - \frac{d}{dx}\left(\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz}\right) = 4\pi \frac{du}{dt},$$

where (§ 127) for an isotropic medium with magnetic rotatory quality, and for periodic disturbances of type $e^{\iota pt}$,

$$4\pi c^2 \frac{du}{dt} = \frac{d^2}{dt^2}\left(KP + a_3 \frac{dQ}{dt} - a_2 \frac{dR}{dt}\right)$$

$$= K\frac{d^2P}{dt^2} - p^2 a_3 \frac{dQ}{dt} + p^2 a_2 \frac{dR}{dt}.$$

Thus if the system is referred to new axes rotating around the axis of the magnetic field with an angular velocity

$$\tfrac{1}{2}\frac{p^2}{K}(a_1,\ a_2,\ a_3)$$

whose square is negligible, we shall have

$$4\pi c^2\frac{du}{dt}=K\frac{d^2P}{dt^2},$$

and the equations will be those of the same isotropic medium but with the magnetic influence absent. The effect of the magnetic field on *any* periodic oscillation or wave-train is therefore to cause rotation with angular velocity $\tfrac{1}{2}\frac{p^2}{K}$ $(a_1,\ a_2,\ a_3)$ around its axis. In the case of a plane wave-train inclined to the axis, the rotation in the plane of the wave-front, which is equal to the rotation of the polarized vibration per unit length of the medium multiplied by its velocity of propagation, is proportional to the component of the magnetic field at right angles to the wave-front; while the other component only displaces the wave-front sideways without changing its direction, so that the direction of propagation is not altered.

This statement will also hold approximately for a crystalline medium provided the differences between its principal indices of refraction are small. Then the radiant vibration which is being propagated in the manner determined by the crystalline quality is at the same time gradually turning round in its plane with the angular velocity here determined, whose value depends on the direction of the wave-front. It is a question of simple kinematics to find the velocities and the elliptic polarizations* of the wave-trains that will be propagated, under these superposed influences, without change of type.

For structural rotation (§ 133) in an isotropic medium

$$4\pi c^2\frac{du}{dt}=\frac{d^2}{dt^2}\left(KP+A\frac{dQ}{dz}-A\frac{dR}{dy}\right),$$

* Cf. Gouy, *Journ. de Phys.* 1885; Lefebvre, *Journ. de Phys.* 1892; O. Wiener, *Wied. Ann.* 1888.

which, for a wave-train of type $\exp \iota (lkx + mky + nkz - pt)$ for which $k = 2\pi/\lambda$, becomes

$$4\pi c^2 \frac{du}{dt} = K \frac{d^2P}{dt^2} + Akpn \frac{dQ}{dt} - Akpm \frac{dR}{dt};$$

so that the effect is represented by a rotation of the disturbance suitable for a non-rotational medium, with angular velocity equal to $-\frac{1}{2} \frac{pk}{K} A (l, m, n)$ and therefore around the direction of propagation. In a crystalline medium K will be replaced by (K_1, K_2, K_3), and when the principal rotational axes coincide with those of the double refraction A will be simply replaced by (A_1, A_2, A_3): thus when the differences of the principal indices are small, the effect of the rotational terms will be equivalent to an imposed angular velocity $-\frac{1}{2} \frac{pk}{K} (A_1 l, A_2 m, A_3 n)$*. For a uniaxial crystal like quartz the vector (A_1, A_2, A_3) must be directed along the axis: thus the coefficient of effective rotation, that is of the component around the normal to the wave-front, is proportional to the square of the cosine of its inclination to the axis of the crystal.

* The determination of the circumstances of the wave-trains of permanent type, directly from the equations, for the special rotational scheme here given, has been effected by Goldhammer, *Journ. de Phys.* 1892. A solution has also been given by him for the problem of refraction into a rotationally active medium, by satisfying all the ordinary boundary conditions for the electric vectors. This latter question has however been referred to (pp. 207, 329) as an instance in which the transition at the boundary may not legitimately be treated as abrupt: the Action function now involves spacial differentiations higher than the first, and it is from its variation that the type of the rotational quality and the dynamical equations are both ultimately determined: as regards the latter, there occur boundary terms involving the independent variations of the first gradients as well as those of the variables themselves, and these though small are of the order of the rotational effect, and will not be annulled by the procedure above mentioned.

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